Value-at-Risk-Based Risk Management Using Options

Cheng Zhang* Yang Zhou† Zhiping Zhou‡

January 14, 2013

Abstract

This paper investigates optimal portfolio and wealth strategy of an institutional investor under the Value-at-Risk (VaR) constraint in an economy under jump diffusion. We show that overlooking or underestimating jump risk factor could be the cause of failure to satisfy the VaR constraint in the recent financial crisis for many financial institutions. We also find that the introduction of the jump risk factor drives the institutional investor to move towards the portfolio insurance strategy, alleviating the problem with VaR identified by Basak and Shapiro (2001) that VaR risk manager incurs larger losses than non risk manager in worst scenarios.

Keywords: Asset Allocation, Derivatives, Value-at-Risk, Risk Management

*Department of Finance, London School of Economics and Political Science, London WC2A 2AE, Email: c.zhang14@lse.ac.uk

†Department of Finance, Tilburg University, Tilburg 5037 DB, Email: y.zhou@uvt.nl

‡Department of Finance, Bocconi University, Milan 20142, Email: zhiping.zhou@phd.unibocconi.it

1 Introduction

In this paper, we study optimal portfolio and wealth policies of an institutional investor maximizing her utility and facing the Value-at-Risk (VaR) constraint in an economy under jump diffusion. Specifically, the investor can access not only the bond and stock markets but also to the derivative market to manage the two risk exposures, namely the jump risk exposure and the diffusion risk exposure, under the VaR constraint.

The past few years have witnessed a global financial meltdown, which raised a heated discussion on the massive failures of risk measurement and management in the financial industry. Due to its practical advantages, VaR is widely used by financial as well as nonfinancial firms as a tool to manage and control risk (Basak and Shapiro (2001)). As outlined by the Finance Professor Rene Stulz, both ignoring known risk and devising the wrong response to risk can lead to financial mismanagement. Therefore, it is highly likely that the failures occurring in the recent financial crisis are stemming from overlooking some major risk factors such as the jump risk and using the risk measures in an inappropriate way. Basak and Shapiro (2001) analyze a dynamic optimization problem of the VaR agent only facing the diffusion risk. It is therefore natural to extend their work by incorporating jump risk into the VaR risk management framework.

Our paper builds on the literature on options investment and pricing. Liu and Pan (2003) study a portfolio choice problem with both jump risk and stochastic volatility and highlight the role of the options in investment from the speculative point of view. Tan (2009) analyzes the attractiveness of European style call and put options for long horizon investors. In addition, most articles are focusing on the market incompleteness and the option pricing (see, for example, Zhang, Zhao, and Chang (2012) and Kou (2002)). On the other hand, a variety of empirical works estimate the jump risk premium embedded in the options (see, for example, Bates (2000) and Pan (2002)). While it is well documented how the derivatives are used to span the market and how price jumps affect the asset allocation, few papers emphasize the hedging roles of derivatives in a proper context. Ahn, Boudoukh, Richardson, and Whitelaw

(1999) is one of exceptions to analyze the roles of options in minimizing the firms' VaR. However, they solve the problem in the Black Scholes world in which the options are redundant, and only explore the usage of options from the perspective of risk management. In contrast, we intend to examine asset allocation problem by employing options to manage the jump risk in a VaR-framework and highlight the hedging roles of options.

Our paper also builds on the literature studying the dynamic optimization problem under risk measure constraints in a variety of settings. Basak and Shapiro (2001) study the institutional investors' portfolio construction and wealth polices subject to a single-VaR constraint and a single Limited-Expected-Losses (LEL) constraint, and find that the presence of VaR-type investor magnifies the market volatility by inducing increased risk exposure in "bad states". By allowing for dynamically reevaluated VaR limit, Cuoco, He, and Isaenko (2008) find that a VaR limit does not necessarily generate negative impact and actually serves as an appropriate tool in managing risk. Shi and Werker (2012) extend Cuoco, He, and Isaenko (2008) by imposing repeated short-horizon VaR limits and allowing for rebalancing of the portfolio between two VaR reviews and examine the effects of the misalignment between investment and regulation horizons on the portfolio strategy of the institutional investors. Different from the extant literature, however, our paper is the first to study the effects of jump risk on the optimal portfolio strategy under the VaR constraint.

The rest of the paper is organized as follows.

2 The Model

We consider a complete financial market with a finite horizon $[0, T]$. In this market, three securities are available: a riskless bond paying a constant interest rate r, a risky stock representing the aggregate equity market and a derivative security based on the stock. Denote by B, S, O the price processes of the bond, the stock and the derivative security. The dynamics for the price processes of the bond and the stock are assumed as follows:

$$\frac{\mathrm{d}B_t}{B_{t-}} = r\mathrm{d}t, \tag{1}$$

$$\frac{\mathrm{d}S_t}{S_{t-}} = \left(r + \eta\sigma + \mu\left(\lambda - \lambda^Q\right)\right)\mathrm{d}t + \sigma\mathrm{d}Z_t + \mu\left(\mathrm{d}N_t - \lambda\mathrm{d}t\right), \tag{2}$$

where Z is a standard Brownian motion and N is a pure-jump process. They are assumed to be independent of each other. σ is the equity market volatility. μ and λ are the jump size and jump arrival intensity associated with the pure-jump process N. Following Liu and Pan (2003), we assume constant jump sizes, which imply that conditional on a jump arrival, the stock price jumps by a constant multiple of $\mu > -1$, with the limiting case of -1 capturing total loss. The benefit of this simple specification is that only one extra derivative security is needed to complete the market with respect to the jump risk. Furthermore, we don't incorporate stochastic volatility and jump in volatility in our model, which are common in the literature on options (see, for example, Liu and Pan (2003) and Branger, Schlag, and Schneider (2008)), because our focus is not on the empirical properties of options but rather on the the role of options in hedging jump risk under the framework of the VaR risk management. Finally, η and λ^Q capture the two components of the equity premium: one is for diffusive risk Z and the other for jump risk N.

The price the derivative security is assumed to follow:

$$\frac{\mathrm{d}O_t}{O_{t-}} = \left(r + \eta\frac{g_S}{O_t}S_t\sigma + \frac{\Delta g}{O_t}\left(\lambda - \lambda^Q\right)\right)\mathrm{d}t + \frac{g_S}{O_{t-}}S_t\sigma\mathrm{d}Z_t + \frac{\Delta g}{O_{t-}}\left(\mathrm{d}N_t - \lambda\mathrm{d}t\right) \tag{3}$$

where g_S measures the sensitivity of the derivative price to the infinitesimal changes in the stock price and Δg measures the change in the derivative price for each stock price jump. Specifically,

$$g_S = \frac{\partial g(S)}{\partial S}; \qquad \Delta g = g((1+\mu)S_t) - g(S_t) \tag{4}$$

The self-financing condition implies that the wealth process evolves as,

$$\frac{\mathrm{d}W_t}{W_{t-}} = \left(r + \pi_t^Z \eta\sigma + \pi_{t-}^N \mu \left(\lambda - \lambda^Q\right)\right) \mathrm{d}t + \pi_t^Z \sigma \mathrm{d}Z_t + \pi_{t-}^N \mu (\mathrm{d}N_t - \lambda \mathrm{d}t), \tag{5}$$

where π_t^Z and π_t^N capture the exposure of the wealth process to diffusive risk and jump risk respectively. Denote by x^S and x^O the fractions of wealth invested in the stock and derivative security. Then, π_t^Z and π_t^N are defined by,

$$\pi_t^Z = x_t^S + x_t^O \frac{g_S S_t}{O_t}, \tag{6}$$

$$\pi_t^N = x_t^S + x_t^O \frac{\Delta g}{\mu O_t}. \tag{7}$$

The interpretation of (6) and (7) is that by investing x^S of the wealth in the stock and x^O in the derivative security amounts to investing π_t^Z in the diffusive risk factor Z, π_t^N in the jump risk factor N. It is important to realize that to complete the market with respect to jump risk, the derivative security must have different sensitivities to infinitesimal and large changes in stock prices: $\frac{g_S S_t}{O_t} \neq \frac{\Delta g}{\mu O_t}$.

In the complete market described above, there exits a unique pricing kernel ξ, whose dynamics is,

$$\frac{\mathrm{d}\xi_t}{\xi_{t-}} = -r\mathrm{d}t - \eta \mathrm{d}Z_t - \left(1 - \frac{\lambda^Q}{\lambda}\right)(\mathrm{d}N_t - \lambda \mathrm{d}t) \tag{8}$$

where $\xi_0 = 1$. As we focus on negative jumps $\mu < 0$, the market price of jump risk must be negative, which mandates $\lambda^Q > \lambda$. It is important to recognize that the introduction of the jump risk may lead to major variations in the distribution of ξ, which are likely to have substantial impact on the risk management and measurement. Note that when there is no jump risk premium ($\lambda^Q = \lambda$), the third term in (8) drops out and (8) reduces to pricing kernel in the absence of jump risk.

3 Optimization under the VaR constraint

3.1 Optimal Portfolio Wealth

We consider an institutional investor, who is initially endowed with wealth of W_0 and is concerned with maximizing the expected utility over the terminal wealth. We assume that the institutional investor has CRRA preferences with relative risk aversion of γ and a fixed investment horizon of T.

The institutional investor is subject to a constraint of a VaR type at the horizon imposed by the regulator, which can be formulated as

$$P(W_T \leq \underline{W}) \leq \alpha, \tag{9}$$

where the "floor" $\underline{W}$ and the loss probability α are specified exogenously by the regulator. The VaR constraint requires that the probability that the institutional investor's wealth at the horizon falls below the floor wealth $\underline{W}$ be α or less. Following Basak and Shapiro (2001), We also consider two alternative cases: one is the benchmark case (B), in which the VaR constraint is never binding and the other one is the portfolio insurance case (PI), in which the horizon wealth is constrained to be above the floor $\underline{W}$ in all states. Note that (9) nests both the B-case and the PI-case, which correspond to $\alpha = 1$ and $\alpha = 0$ respectively. Therefore, the VaR-case can be thought of as an intermediate case between the two extreme cases, the B-case and the PI-case.

The portfolio optimization problem of the VaR agent can be formulated as follows,

$$\max_{W_T} \mathrm{E}\left[\frac{W_T^{1-\gamma}}{1-\gamma}\right] \tag{10}$$

$$\text{subject to} \quad \mathrm{E}[\xi_T W_T] \leqslant \xi_0 W_0 \tag{11}$$

$$P(W_T \leqslant \underline{W}) \leqslant \alpha. \tag{12}$$

Following Basak and Shapiro (2001), we solve this problem using the martingale representation approach. Proposition 1 characterizes the optimal terminal wealth under the VaR constraint.

Proposition 1. *The time-T optimal wealth of the VaR agent is*

$$W_T^{VaR} = \begin{cases} \left(y^{VaR}\xi_T\right)^{-\frac{1}{\gamma}} & \text{if } \xi_T < \underline{\xi}^{VaR}, \\ \underline{W} & \text{if } \underline{\xi}^{VaR} \leqslant \xi_T < \bar{\xi}^{VaR}, \\ \left(y^{VaR}\xi_T\right)^{-\frac{1}{\gamma}} & \text{if } \xi_T \geqslant \bar{\xi}^{VaR}. \end{cases} \tag{13}$$

where $\underline{\xi}^{VaR} = \frac{\underline{W}^{-\gamma}}{y^{VaR}}$, $\bar{\xi}^{VaR}$ *is such that* $P(\xi_T > \bar{\xi}^{VaR}) = \alpha$, *and* y^{VaR} *is the Lagrange multiplier of the budget constraint and solves* $\mathrm{E}\left[\xi_T W_T^{VaR}\right] = \xi_0 W_0$. *The VaR constraint is binding if and only if* $\underline{\xi}^{VaR} < \bar{\xi}^{VaR}$. *Moreover,* $y^{VaR} \in \left[y^B, y^{PI}\right]$.

Proposition 1 indicates that if the VaR constraint is binding, the VaR agent's optimal horizon wealth is classified into three distinct regions: in both regions of "good states" $\left[\xi_T < \underline{\xi}^{VaR}\right]$ and "bad states" $\left[\xi_T \geqslant \bar{\xi}^{VaR}\right]$, her terminal wealth is decreasing in ξ_T, while in the region of "intermediate states" $\left[\underline{\xi}^{VaR} \leqslant \xi_T < \bar{\xi}^{VaR}\right]$ her terminal wealth is kept constant at the portfolio insurance level. The definition of the upper bound $\bar{\xi}^{VaR}$ implies that the probability of the bad states region stays constant at α. Moreover, $y^{VaR} \in \left[y^B, y^{PI}\right]$ confirms that the VaR case is intermediate between the B case and the PI case.

Basak and Shapiro (2001) show that the VaR agent's optimal terminal wealth in (13) can be decomposed as

$$\begin{aligned} W_T^{VaR}(y(W_0)) &= W_T^{PI}(y^B(W_*)) - (\underline{W} - W_T^B(y^B(W_*)))1_{\{\xi_T \geqslant \bar{\xi}\}} & (14) \\ &= W_T^B(y^B(W_*)) + (\underline{W} - W_T^B(y^B(W_*)))1_{\{\underline{\xi} \leqslant \xi_T < \bar{\xi}\}}, & (15) \end{aligned}$$

where W_* is set so that $y^B(W_*) = y(W_0)$. Put differently, W^{VaR} is equivalent to a PI solution plus a short position in "binary" options, or a B solution plus an appropriate position in "corridor" options.

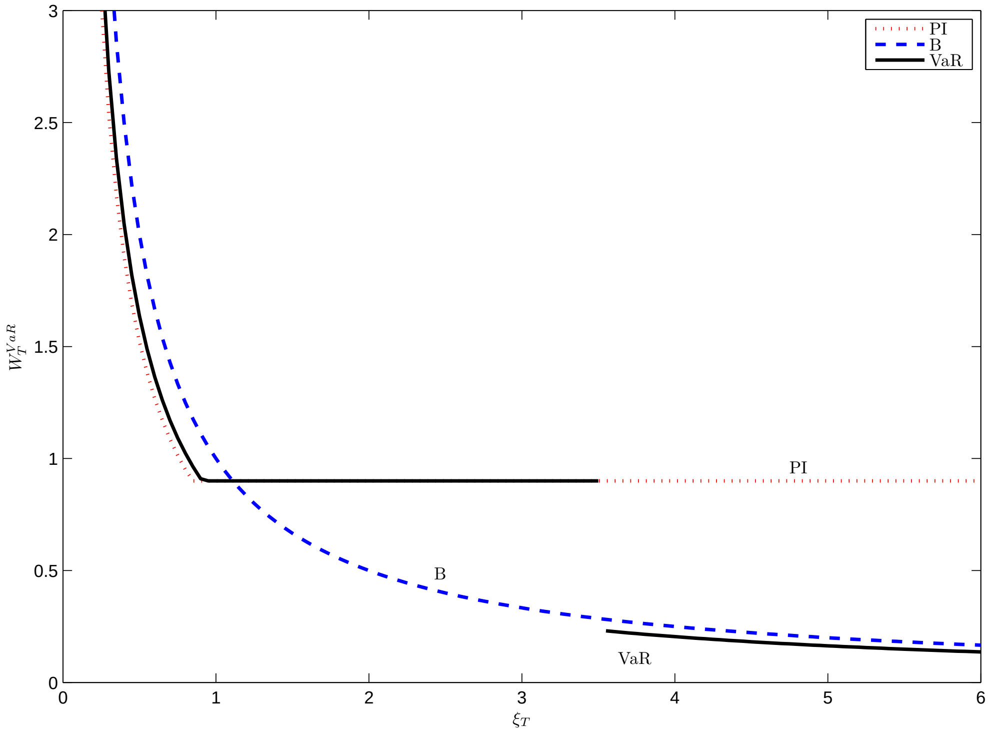

Figure 1: Optimal horizon wealth of three types of agents. The figure plots the optimal horizon wealth of the VaR agent (solid line), the benchmark agent (dashed line) and the portfolio insurance agent (dotted line). The parameter values are: $\lambda = 1$, $\lambda^Q = 1.5$, $\xi_0 = 1$, $r = 0.05$, $\eta = 0.4$, $\sigma_S = 0.18$, $T = 1$, $t = 0.5$, $\gamma = 1$, $\alpha = 0.01$, $W_0 = 1$, $\underline{W} = 0.9$.

Figure 1 illustrates the optimal terminal wealth of a VaR agent, a benchmark agent $(\alpha = 1)$ and a portfolio insurance agent $(\alpha = 0)$. Consistent with Proposition 1, in both regions of good states and bad states, the VaR agent behaves like the B agent. In contrast, in the intermediate states she adopts portfolio insurance strategy as the PI agent does. A striking feature of the VaR agent's horizon wealth strategy is that she leaves the bad states fully uninsured, as they are most costly to insure against; her wealth is even lower than the B agent's wealth in the worst state for any given ξ_T. In other words, the VaR agent ignores losses in the upper tail of the ξ_T distribution, which is independent of the agent's preferences and endowment but dependent on the jump risk premium.

Figure 2 illustrates the distribution of the pricing kernel at horizon for different jump parameters. Note that $\lambda^Q/\lambda = 1$ corresponds to zero jump risk premium and reduces to the Basak and Shapiro case. Obviously, as the jump risk premium increases, the distribution of ξ_T

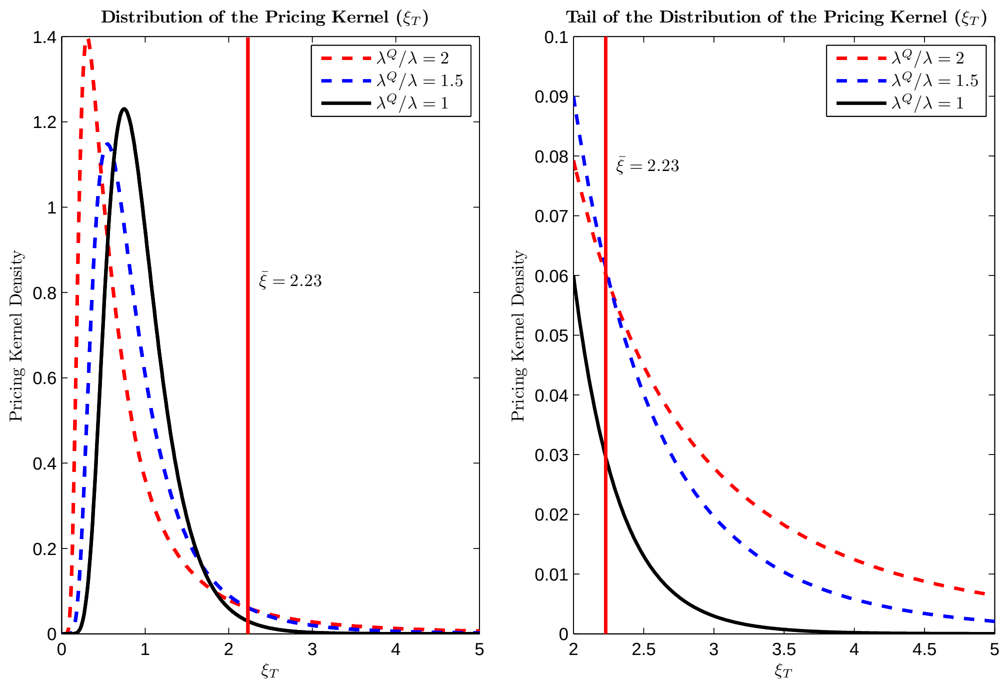

Figure 2: Distribution of the pricing kernel at horizon. The left plots the distribution of the pricing kernel at horizon for different jump parameters, while the right panel plots the upper tail of the distribution. In both panels, the black solid line is for $\lambda^Q/\lambda = 1$, the blue dashed line is for $\lambda^Q/\lambda = 1.5$ and the red dashed line is for $\lambda^Q/\lambda = 2$. The red solid line is for $\bar{\xi}$ in the Basak and Shapiro case. λ is fixed at 1, while λ^Q varies in different cases. Other parameter values are: $\xi_0 = 1$, $r = 0.05$, $\eta = 0.4$, $T = 1$.

moves leftwards, while the upper tail of the distribution gets fat. The detrimental consequence of these variations is that the probability beyond the upper bound of the states $\bar{\xi}^{VaR}$ implied by Basak and Shapiro (2001) exceeds the pre-specified loss probability α, thereby leading to violation of the VaR constraints. This observation is more clearly confirmed by the right panel, which plots the upper tail of the distribution of ξ_T. In other words, if an institutional investor follows the terminal wealth distribution proposed by Basak and Shapiro (2001) in a market with positive jump risk premium, she will choose a larger region as bad states, which she leaves completely uninsured, than what it is supposed to be as implied by the distribution of ξ_T and make too aggressive investment decisions. Therefore, overlooking jump risk could be a cause of failure to satisfy the VaR constraints in the recent financial crisis for many institutional investors. Therefore, although Proposition 1 is almost identical to the solution in Basak and Shapiro (2001) except for the specific CRRA utility function, fundamental differ-

Table 1: Classification of States for Optimal Horizon Wealth Under the VaR Constraint

λ	λ^Q	$\underline{\xi}^{VaR}$	$\bar{\xi}^{VaR}$	Pr^{VaR}	Pr^{PI}	y^{VaR}	y^B	y^{PI}
1	1	0.99	2.23	37.0%	40.5%	1.12	1	1.15
1	1.5	0.91	3.51	37.7%	43.6%	1.22	1	1.31
1	2	0.83	5.98	33.1%	42.2%	1.34	1	1.60
1	3	0.82	10.38	21.4%	26.5%	1.35	1	1.69
1.5	2	0.93	3.12	38.5%	43.7%	1.20	1	1.27
1.5	3	0.80	7.27	30.8%	40.1%	1.39	1	1.74
2	3	0.86	4.50	37.3%	44.8%	1.30	1	1.46

The table shows the classification of the state of the world for optimal horizon wealth under the VaR constraint. Pr^{VaR} is the probability that the VaR agent's terminal wealth is under portfolio insurance. Pr^{PI} is the probability that the PI agent's terminal wealth is under portfolio insurance. The parameter values are: $\xi_0 = 1$, $r = 0.05$, $\eta = 0.4$, $T = 1$.

ences exist between our jump case and the Basak and Shapiro case due to the leftward shift in the distribution of the pricing kernel.

Table 1 reports the classification of states for optimal horizon wealth under the VaR constraint for different jump parameters. The upper bound $\bar{\xi}^{VaR}$ is decreasing in λ for any given λ^Q, but increasing in λ^Q for any given λ. In contrast, the opposite holds for the lower bound $\underline{\xi}^{VaR}$. As a consequence, the region of the intermediate states widens as the jump risk premium increases. These results are obviously consistent with Figure 1: as the tail gets fat, $\bar{\xi}^{VaR}$ must increase to make sure that the probability beyond $\bar{\xi}^{VaR}$ is equal to α. On the other hand, $\underline{\xi}^{VaR}$ must decrease to satisfy the budget constraint due to the leftward shift in the distribution of ξ_T. While the probability in the region of worst states remains constant at α as prescribed by the VaR constraint, the probability in each of the two regions varies across different jump risk premiums. Since the VaR agent chooses to fully insure against the intermediate states, one can think of the probability in the region of the intermediate states as the cost of satisfying the VaR constraint. In Table 1, we denote the probability that the terminal wealth is under portfolio insurance for the VaR agent by Pr^{VaR} and that for the PI agent by Pr^{PI}. It is revealed that in general, both Pr^{VaR} and Pr^{PI} decrease with the jump risk premium, indicating lower costs of meeting the VaR constraints. This can be explained by the fact that although both $\underline{\xi}^{VaR}$ and the whole distribution of ξ_T shift to the left due to the jump premium, the latter one shifts more sharply when λ is sufficiently large, rendering the probability under portfolio insurance to shrink. In addition, while the Lagrange multiplier in the B case keeps constant, the Lagrange multiplier in both the VaR case and the PI case increases with the jump risk premium, implying that the introduction of jump risk premium induces both the VaR and the PI agents to deviate more from the benchmark case. **How to interpret the wider gap between y^{VaR} and y^{PI}?**

3.2 Trading Strategies

Proposition 2 characterizes the optimal wealth and portfolio strategies before the horizon under the VaR constraint.

Proposition 2. *The time-t optimal wealth is given by*

$$\begin{aligned} W_t^{VaR} = \frac{\mathrm{e}^{\Gamma_t}}{(y\xi_t)^{\frac{1}{\gamma}}} - \Bigg\{ \frac{\mathrm{e}^{\Gamma_t}}{(y\xi_t)^{\frac{1}{\gamma}}} \mathrm{E}_t \left[\mathrm{e}^{\frac{1-\gamma}{\gamma}\Psi(N_t)} \mathcal{N}\left(-d_1(\underline{\xi}^{VaR}) - \frac{\Psi(N_t)}{\eta\sqrt{\tau}} \right) \right] \\ - \underline{W}\mathrm{e}^{-r\tau}\mathrm{E}_t \left[\mathrm{e}^{-\Psi(N_t)} \mathcal{N}\left(-d_2(\underline{\xi}^{VaR}) - \frac{\Psi(N_t)}{\eta\sqrt{\tau}} \right) \right] \Bigg\} \\ + \Bigg\{ \frac{\mathrm{e}^{\Gamma_t}}{(y\xi_t)^{\frac{1}{\gamma}}} \mathrm{E}_t \left[\mathrm{e}^{\frac{1-\gamma}{\gamma}\Psi(N_t)} \mathcal{N}\left(-d_1(\bar{\xi}^{VaR}) - \frac{\Psi(N_t)}{\eta\sqrt{\tau}} \right) \right] \\ - \underline{W}\mathrm{e}^{-r\tau}\mathrm{E}_t \left[\mathrm{e}^{-\Psi(N_t)} \mathcal{N}\left(-d_2(\bar{\xi}^{VaR}) - \frac{\Psi(N_t)}{\eta\sqrt{\tau}} \right) \right] \Bigg\} \end{aligned} \qquad (16)$$

where $\tau = T - t$, y is given in Proposition 1, $\mathcal{N}(\cdot)$ is the standard-normal cumulative distri-

bution function and

$$\underline{\xi}^{VaR} = \frac{1}{y\underline{W}^{\gamma}},$$
$$\Gamma(t) = \frac{1-\gamma}{\gamma}\left(r + \frac{\eta^2}{2}\right)\tau + \left(\frac{1-\gamma}{\gamma}\right)^2 \frac{\eta^2}{2}\tau,$$
$$\Psi(N_t) = (\lambda^Q - \lambda)\tau - \ln\left(\frac{\lambda^Q}{\lambda}\right)(N_T - N_t)$$
$$d_2(x) = \frac{\ln\left(\frac{x}{\xi_t}\right) + \left(r - \frac{\eta^2}{2}\right)\tau}{\eta\sqrt{\tau}},$$
$$d_1(x) = d_2(x) + \frac{\eta}{\gamma}\sqrt{\tau}.$$

Note that all of the expectations in (16) are taken with respect to $(N_T - N_t)$*.*

The exposure of the optimal portfolio to the risk factors Z *and* N *is given by,*

$$\begin{aligned}\pi_t^{Z,VaR} = \frac{\eta}{\sigma\gamma} &- \frac{e^{-r\tau}\eta\underline{W}}{\sigma\gamma W_t^{VaR}} E_t\left[e^{-\Psi(N_t)}\left(\mathcal{N}\left(-d_2(\underline{\xi}^{VaR}) - \frac{\Psi(N_t)}{\eta\sqrt{\tau}}\right) - \mathcal{N}\left(-d_2(\bar{\xi}^{VaR}) - \frac{\Psi(N_t)}{\eta\sqrt{\tau}}\right)\right)\right] \\ &+ \frac{e^{-r\tau}\underline{W}}{\sigma\sqrt{\tau}W_t^{VaR}} E_t\left[e^{-\Psi(N_t)}\left(\phi\left(d_2(\bar{\xi}^{VaR}) + \frac{\Psi(N_t)}{\eta\sqrt{\tau}}\right) - \phi\left(d_2(\underline{\xi}^{VaR}) + \frac{\Psi(N_t)}{\eta\sqrt{\tau}}\right)\right)\right] \\ &- \frac{e^{\Gamma_t}(y\xi_t)^{-\frac{1}{\gamma}}}{\sigma\sqrt{\tau}W_t^{VaR}} E_t\left[e^{\frac{1-\gamma}{\gamma}\Psi(N_t)}\left(\phi\left(d_1(\bar{\xi}^{VaR}) + \frac{\Psi(N_t)}{\eta\sqrt{\tau}}\right) - \phi\left(d_1(\underline{\xi}^{VaR}) + \frac{\Psi(N_t)}{\eta\sqrt{\tau}}\right)\right)\right],\end{aligned} \tag{17}$$

$$\pi_t^{N,VaR} = \frac{\sigma\left(1 - \frac{\lambda^Q}{\lambda}\right)}{\mu\eta}\pi_t^{Z,VaR}. \tag{18}$$

In the benchmark case, the exposure of the optimal portfolio to the risk factor Z *is,*

$$\pi_t^B = \frac{\eta}{\gamma\sigma} \tag{19}$$

Let $q_t^{VaR} = \pi_t^{Z,VaR}/\pi_t^B$. Then q^{VaR} is,

$$
\begin{aligned}
q_t^{VaR} = 1 &- \frac{e^{-r\tau}\underline{W}}{W_t^{VaR}} E_t \left[e^{-\Psi(N_t)} \left(\mathcal{N}\left(-d_2(\underline{\xi}^{VaR}) - \frac{\Psi(N_t)}{\eta\sqrt{\tau}} \right) - \mathcal{N}\left(-d_2(\bar{\xi}^{VaR}) - \frac{\Psi(N_t)}{\eta\sqrt{\tau}} \right) \right) \right] \\
&+ \frac{\gamma e^{-r\tau}\underline{W}}{\eta\sqrt{\tau}W_t^{VaR}} E_t \left[e^{-\Psi(N_t)} \left(\phi\left(d_2(\bar{\xi}^{VaR}) + \frac{\Psi(N_t)}{\eta\sqrt{\tau}} \right) - \phi\left(d_2(\underline{\xi}^{VaR}) + \frac{\Psi(N_t)}{\eta\sqrt{\tau}} \right) \right) \right] \\
&- \frac{\gamma e^{\Gamma_t}(y\xi_t)^{-\frac{1}{\gamma}}}{\eta\sqrt{\tau}W_t^{VaR}} E_t \left[e^{\frac{1-\gamma}{\gamma}\Psi(N_t)} \left(\phi\left(d_1(\bar{\xi}^{VaR}) + \frac{\Psi(N_t)}{\eta\sqrt{\tau}} \right) - \phi\left(d_1(\underline{\xi}^{VaR}) + \frac{\Psi(N_t)}{\eta\sqrt{\tau}} \right) \right) \right].
\end{aligned} \tag{20}
$$

Transforming the π's to the optimal portfolio weights on the stock x_t^S and the option x_t^O, we have

$$
x_t^S = \left(1 - \left(\frac{\Delta g}{\mu O_t} - \frac{g_S S_t}{O_t} \right)^{-1} \frac{g_S S_t}{O_t} \left(\frac{\sigma\left(1 - \frac{\lambda^Q}{\lambda}\right)}{\mu\eta} - 1 \right) \right) \pi_t^{Z,VaR} \tag{21}
$$

$$
x_t^O = \left(\frac{\Delta g}{\mu O_t} - \frac{g_S S_t}{O_t} \right)^{-1} \left(\frac{\sigma\left(1 - \frac{\lambda^Q}{\lambda}\right)}{\mu\eta} - 1 \right) \pi_t^{Z,VaR} \tag{22}
$$

Once again, consistent with Basak and Shapiro (2001), (16) reveals that the optimal time-t wealth consists of three components: a myopic component that maximizes Sharpe ratio and represents the optimal wealth of the B agent and two option components that correspond to **a long position in a put option on the B agent's wealth with strike price of $\underline{W}$ and a short position in a binary option on the same security with the same strike price.** However, in contrast to Basak and Shapiro (2001), the option prices do not immediately follow from the Black-Scholes option pricing formula, but rather are computed as the expectation of option prices conditional on jumps realized with respect to the jump risk factor.

Figure 3 depicts the optimal wealth of the VaR agent, the B agent and the PI agent at time t. The optimal prehorizon wealth of the VaR agent behaves similarly to that of the B agent in both the good and bad states. In contrast, in the intermediate region, the VaR agent's wealth does not coincide with the PI agent's wealth, because she just begins to insure against

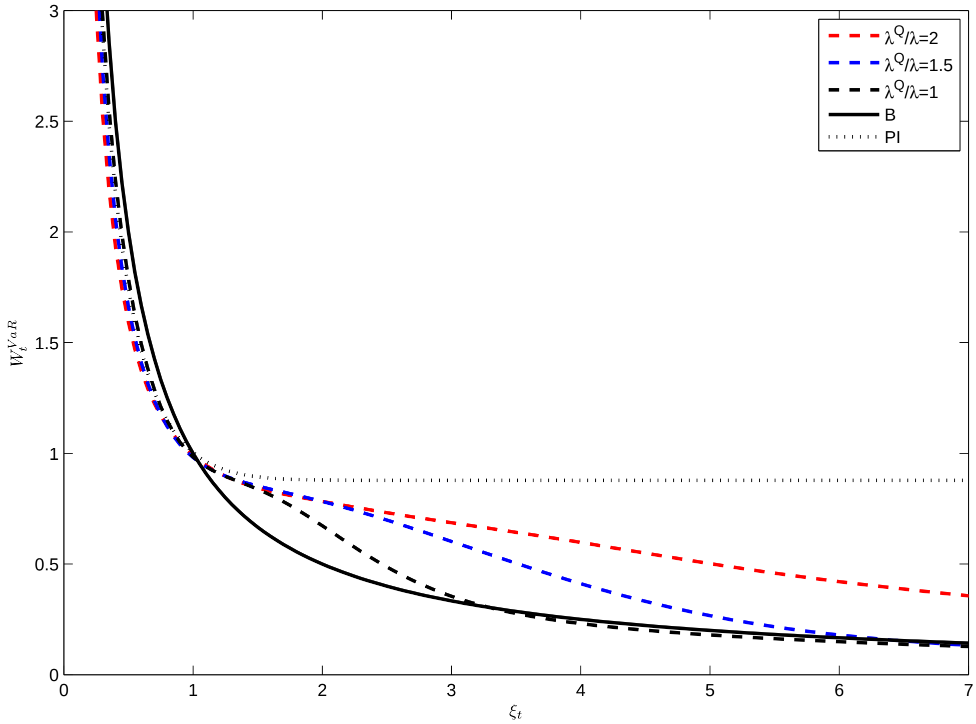

Figure 3: Optimal time-t wealth of different types of agents. The figure plots the optimal time-t wealth of the VaR agent, the benchmark agent and the portfolio insurance agent. The black solid line is for $\lambda^Q/\lambda = 1$, the blue dashed line is for $\lambda^Q/\lambda = 1.5$ and the red dashed line is for $\lambda^Q/\lambda = 2$. The solid line is for the benchmark case and the dotted line is for the portfolio insurance case. λ is fixed at 1, while λ^Q varies in different cases. Other parameter values are: $\xi_0 = 1$, $r = 0.05$, $\eta = 0.4$, $\sigma_S = 18$, $T = 1$, $t = 0.5$, $\gamma = 1$, $\alpha = 0.01$, $W_0 = 1$, $\underline{W} = 0.9$.

intermediate state. An interesting observation is that as the jump risk premium increases, the VaR agent enjoys higher wealth in the intermediate and worst states at the expense of lower wealth in the favorable states. More importantly, her wealth goes above the B agent's wealth in more of bad states, thereby alleviating the problem with VaR identified by Basak and Shapiro (2001) that under the VaR constraints, risk managers optimally take larger exposure to risky assets in unfavorable states and incur larger losses than non-risk managers.

Figure 4 illustrates the optimal time-t equity exposure of the VaR agent, the B agent and the PI agent relative to the B agent's equity exposure. In the two extreme states, the VaR agent acts like the B agent. In between, her equity exposure first moves similarly to the PI agent and decreases with ξ_t. Then, she takes increasingly large equity exposure, as the states worsen. Finally, when ξ_T is sufficiently large, the VaR agent's equity exposure again goes

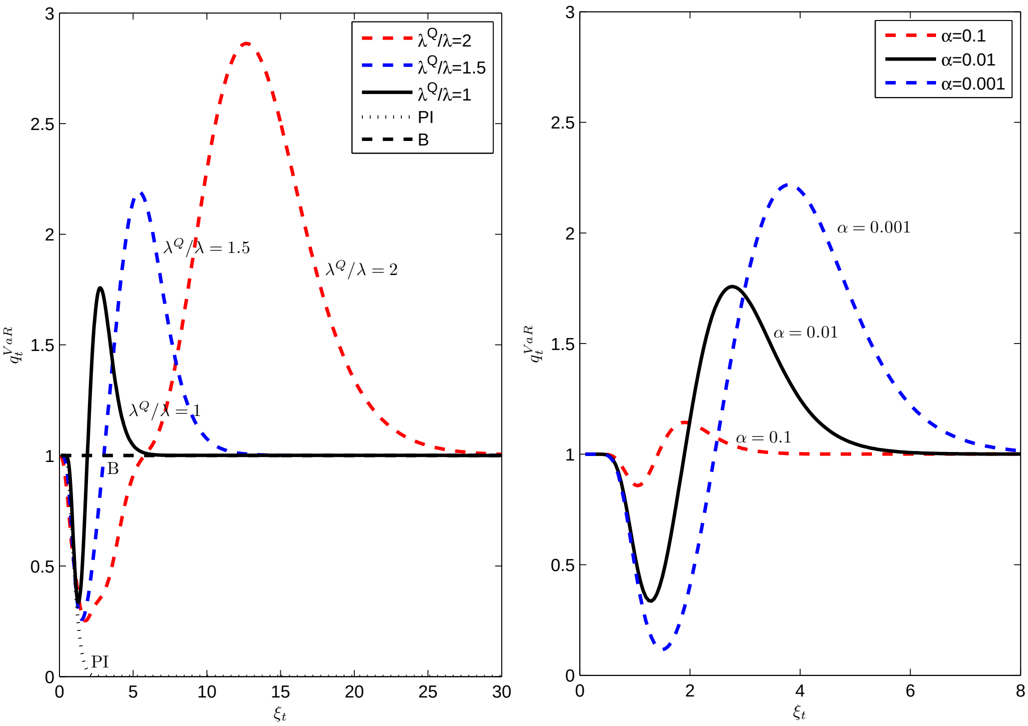

Figure 4: Optimal time-t equity exposure of different types of agents. The left panel plots the optimal time-t equity exposure of the VaR agent, the benchmark agent and the portfolio insurance agent. The right panel plots the optimal pre-horizon risk exposure of the VaR agent in the Basak and Shapiro case for different α for comparison purpose. In the left panel, the solid line is for $\lambda^Q/\lambda = 1$, the blue dashed line is for $\lambda^Q/\lambda = 1.5$ and the red dashed line is for $\lambda^Q/\lambda = 2$. The black dashed line is for the benchmark case and the dotted line is for the portfolio insurance case. In the right panel, the solid line is for $\alpha = 0.01$, the blue dashed line is for $\alpha = 0.001$ and the red dashed line is for $\alpha = 0.1$. λ is fixed at 1, while λ^Q varies in different cases. Other parameter values are: $\xi_0 = 1$, $r = 0.05$, $\eta = 0.4$, $\sigma_S = 18$, $T = 1$, $t = 0.5$, $\gamma = 1$, $\alpha = 0.01$, $W_0 = 1$, $\underline{W} = 0.9$.

back toward the benchmark case. The fluctuation of the VaR agent's equity exposure is due to insuring against the intermediate states: when ξ_t is not so high, she chooses to take a large equity exposure to achieve portfolio insurance level $\underline{W}$. On the contrary, when ξ_t is already very high, all hope is gone and she simply behaves like the B agent. On the other hand, it is obvious that the jump component in the pricing kernel causes the VaR agent to deviate more from the benchmark case and her equity exposure to fluctuate in a larger region. Interestingly, comparison with the Basak and Shapiro case for different α in the right panel reveals that the effect of increasing jump risk premium is similar to that of decreasing α in the benchmark case but is even larger. Therefore, larger jump size of the pricing kernel drives the VaR agent to adjust her asset allocation towards the PI case, making the properties of VaR more desirable from risk management point of view.

(Rewritten part)

Proposition 2 presents the optimal portfolio weights on the stock x_t^S and the option x_t^O. Note that the case of $\mu = 0$ and $\lambda^Q = \lambda = 0$ corresponds to no jump risk framework and reduces to Basak and Shapiro case. In such a setting, the derivative security is redundant,

$$x_t^S = \pi_t^{Z,VaR}, \quad (23)$$

$$x_t^O = 0. \quad (24)$$

Obviously, in a setting with jump risk, the risky stock is affected by two types of risk factors: the diffusive price shock with constant volatility σ, and the jump with Poisson arrival λ and deterministic jump size μ. To complete the market with respect to both the diffusive and jump risks, one derivative security is needed. In such a setting, we will compare the results with that of Liu and Pan (2003), who investigate the optimal portfolio problem under the same market setting for an investor without VaR constraint. Transforming the optimal portfolio weights on the stock x_t^S and the option x_t^O, we have,

$$x_t^S = \pi_t^{Z,VaR} - x_t^O \frac{g_S S_t}{O_t}, \quad (25)$$

$$x_t^O = \left(\frac{\Delta g}{\mu O_t} - \frac{g_S S_t}{O_t} \right)^{-1} (\pi_t^{N,VaR} - \pi_t^{Z,VaR}). \quad (26)$$

Consistent with the results of Liu and Pan (2003), the optimal weight on the option x_t^O is inversely proportional to its ability to disentangle the two exposures. For example, if an option is equally sensitive to infinitesimal and large changes in stock prices, that is, $\frac{g_S S_t}{O_t} = \frac{\Delta g}{\mu O_t}$, then it is not effective at all in providing separate exposures to both the diffusive and jump risks. As a result, the derivative security is redundant. On the contrary, if an option is effective in providing separate exposure, that is, $\frac{g_S S_t}{O_t} \neq \frac{\Delta g}{\mu O_t}$, then it is needed to complete the market. The more effective it is in disentangling the two risk factors, the less it is needed, holding other things constant. In contrast to the results of Liu and Pan (2003), the optimal portfolio weight x_t^O on the option depends on the exposure to the diffusive risk $\pi_t^{Z,VaR}$ and to the jump risk $\pi_t^{N,VaR}$ rather than the case without the VaR constraint.

In our economy setting, the risk-and-return tradeoff is the ultimate driving force for investing the option. If the two risk factors are equally attractive, then the option is also redundant. In this case, the relative value of the two coefficients for the premia of two risk factors, $\frac{\lambda^Q}{\lambda}$ and η respectively, is set such that

$$\frac{\lambda^Q}{\lambda} = 1 - \frac{\mu\eta}{\sigma}. \tag{27}$$

From (22), the optimal weight on the option x_t^O is zero. Because the agent finds the diffusive and jump risks equally attractive, her willingness to disentangle them diminishes, leading to a zero option holding as well. However, the empirical evidence from the option market indicates that the coefficient $\frac{\lambda^Q}{\lambda}$ is much higher than $1 - \frac{\mu\eta}{\sigma}$ (see, for example Pan (2002)). Then, when we consider the option and stock ratio,

$$\frac{x_t^S}{x_t^O} = \frac{\left(1 - \left(\frac{\Delta g}{\mu O_t} - \frac{g_S S_t}{O_t}\right)^{-1} \frac{g_S S_t}{O_t} \left(\frac{\sigma\left(1 - \frac{\lambda^Q}{\lambda}\right)}{\mu\eta} - 1\right)\right)}{\left(\frac{\Delta g}{\mu O_t} - \frac{g_S S_t}{O_t}\right)^{-1} \left(\frac{\sigma\left(1 - \frac{\lambda^Q}{\lambda}\right)}{\mu\eta} - 1\right)}, \tag{28}$$

we find that the option can be used by the agent to load more on the jump risk, holding the condition that it can disentangle the two risk factors. As illustrated in Table 2, when the jump risk permia coefficient $\frac{\lambda^Q}{\lambda}$ jumps from 1.5 to 2, the agent would hold more short positions in put options irrespective of moneyness and time to maturity. In addition, the increase in the jump size μ (absolute value) can have an opposite effect on the option weight. [1] In the situation with large, negative jumps, the agent would not hold too much of jump risk regardless of the high jump risk premium.

The quantitative analysis of optimal strategies is presented in Table 2. Here we consider one jump case: $\mu = -10\%$ jumps once every 10 years and a set of jump risk premia $\frac{\lambda^Q}{\lambda}$. We

[1] To save space, we do not report detailed results and corresponding tables here.

investigate the cross-sectional variation of stock option ratio with respect to both moneyness and maturity. As mentioned before, the out-of-the-money (OTM) put options are effective in distinguishing two risk factors. Put differently, the first term in 26 is more negative for deep OTM put options. Then we incorporate one at-the-money (ATM) option and two OTM options into our analysis.

If jump risk is not being compensated ($\frac{\lambda^Q}{\lambda}$=1), the agent simply use risky stocks to obtain the optimal exposure to diffusive risk. As the risky stocks are suffering from the negative jump (here we only consider the adverse jump), the investor has to hold a small amount of put options to hedge such risk (see the second term in 26 for the delta hedging role played by the options). On one hand, one holds relative fewer amounts of put options with respect to moneyness because a much deeper OTM put option is more effective to disentangle two risk exposures. On the other hand, one holds relative more amounts of put options with respective to time-to-maturity as one has to bear more risks in a longer horizon.

Moreover, the stock holding is switching from positive to negative position as the jump risk premium increases, so does the put option. This is a consequence of the fact that jump risk is becoming more attractive relative to diffusive risk and the agent takes short position to earn this premium, which is consistent with a variety of empirical evidence such as Driessen and Maenhout (2007).

4 Optimization under the LEL Constraint

4.1 Optimal Portfolio Wealth

In this section, we investigate the optimal portfolio strategy under the constraint of Limited Expected Losses (LEL), an alternative risk measure to VaR. In contrast to VaR, which penalizes a high probability of the portfolio wealth loss, LEL is aimed at limiting the magnitude

Table 2: Optimal Strategies under the VaR constraint

$\frac{\lambda^Q}{\lambda}$		1 Month ATM	1 Month 5% OTM	1 Month 10%OTM	3 Month ATM	3 Month 5% OTM	3 Month 10%OTM	1 Year ATM	1 Year 5% OTM	1 Year 10%OTM
1	x_t^S	2.43	1.45	1.16	2.97	2.14	1.71	4.23	3.59	3.13
	x_t^O	0.06	0.01	0.00	0.14	0.06	0.03	0.45	0.32	0.24
	$\frac{x_t^S}{x_t^O}$	40.87	110.80	310.00	21.12	32.98	50.97	9.45	11.27	13.29
1.5	x_t^S	-0.77	0.44	0.80	-1.43	-0.41	0.12	-2.95	-2.17	-1.61
	x_t^O	-0.07	-0.02	0.00	-0.17	-0.08	-0.04	-0.54	-0.39	-0.29
	$\frac{x_t^S}{x_t^O}$	10.47	-27.33	-173.20	8.31	5.11	-2.94	5.46	5.63	5.65
2	x_t^S	-3.90	-0.55	0.45	-5.71	-2.89	-1.43	-9.80	-7.69	-6.17
	x_t^O	-0.20	-0.04	-0.01	-0.47	-0.22	-0.11	-1.46	-1.04	-0.78
	$\frac{x_t^S}{x_t^O}$	19.26	12.25	-35.08	12.08	13.19	12.59	6.72	7.37	7.96

This table shows the optimal strategies under the VaR constraint. The parameter values are: $r = 0.05$, $\eta = 0.4$, $\sigma_S = 0.18$.

and is defined as,

$$E\left[\xi_T(\underline{W} - W_T)1_{\{W_T \leqslant \underline{W}\}}\right] \leqslant \epsilon. \tag{29}$$

where ϵ is the maximum wealth loss. The LEL constraint requires that the expected wealth shortfall cannot exceed a pre-specified level ϵ. It is easy to verify that (29) also nests both the B-case and the PI-case, which correspond to $\epsilon = \infty$ and $\epsilon = 0$ respectively.

With LEL as an additional constraint, the portfolio optimization problem becomes,

$$\max_{W_T} E\left[\frac{W_T^{1-\gamma}}{1-\gamma}\right] \tag{30}$$

$$\text{subject to} \quad E[\xi_T W_T] \leqslant \xi_0 W_0, \tag{31}$$

$$E\left[\xi_T(\underline{W} - W_T)1_{\{W_T \leqslant \underline{W}\}}\right] \leqslant \epsilon. \tag{32}$$

Proposition 3 characterizes the optimal terminal wealth under the LEL constraint.

Proposition 3. *The time-T optimal wealth of the LEL agent is*

$$W_T^{LEL} = \begin{cases} \left(y^{LEL}\xi_T\right)^{-\frac{1}{\gamma}} & \text{if } \xi_T < \underline{\xi}^{LEL}, \\ \underline{W} & \text{if } \underline{\xi}^{LEL} \leqslant \xi_T < \bar{\xi}^{LEL}, \\ \left(\left(y^{LEL} - y_1^{LEL}\right)\xi_T\right)^{-\frac{1}{\gamma}} & \text{if } \xi_T \geqslant \bar{\xi}^{LEL}. \end{cases} \tag{33}$$

where $\underline{\xi}^{LEL} = \frac{\underline{W}^{-\gamma}}{y^{LEL}}$, $\bar{\xi}^{LEL} = \frac{\underline{W}^{-\gamma}}{y^{LEL} - y_1^{LEL}}$, *and* $y^{LEL} \geqslant 0$, $y_1^{LEL} \geqslant 0$ *are two Lagrange multipliers of the budget constraint and the LEL constraint respectively and solve the following system:*

$$\begin{cases} E[\xi_T W^{LEL}(T; y^{LEL}, y_1^{LEL})] = \xi_0 W_0 \\ E\left[\xi_T(\underline{W} - W^{LEL}(T; y^{LEL}, y_1^{LEL}))1_{\left\{W^{LEL}(T; y^{LEL}, y_1^{LEL}) \leqslant \underline{W}\right\}}\right] = \epsilon \text{ or } y_1^{LEL} = 0. \end{cases} \tag{34}$$

The LEL constraint is binding if, and only if, $\underline{\xi}^{LEL} < \bar{\xi}^{LEL}$*. Moreover, the Lagrange multiplier* y^{LEL} *is decreasing in* ϵ, $y^{LEL} \in \left[y^B, y^{PI}\right]$, *and* $y^{LEL} - y_1^{LEL} \leqslant y^B$.

Figure 5 shows the terminal wealth distribution for a LEL agent, a benchmark agent ($\epsilon = +\infty$) and a portfolio insurance agent ($\epsilon = 0$). These optimal wealth levels are determined analytically using Proposition 3. Comparison between Figure 1 and Figure 5 reveals that different from the VaR case, the terminal wealth of the LEL agent is larger than the benchmark agent. The striking distinction follows from the fact that while the VaR agent leaves the bad states fully uninsured, the LEL agent still maintains some level of insurance and makes more conservative investment decisions for those states.

Figure 6 plots the distribution of the pricing kernel at horizon for different jump parameters with $\bar{\xi}^{LEL}$ in the Basak and Shapiro case. Obviously, the fatter tail of the distribution of ξ_T induced by the increase in the jump risk premium generates higher probability of the occurrence of bad states and therefore larger losses in those states. As a consequence, the investor cannot satisfy the LEL constraint, if she ignores the jump risk and pursues the terminal wealth distribution in Basak and Shapiro (2001). This is consistent with the findings in the VaR case

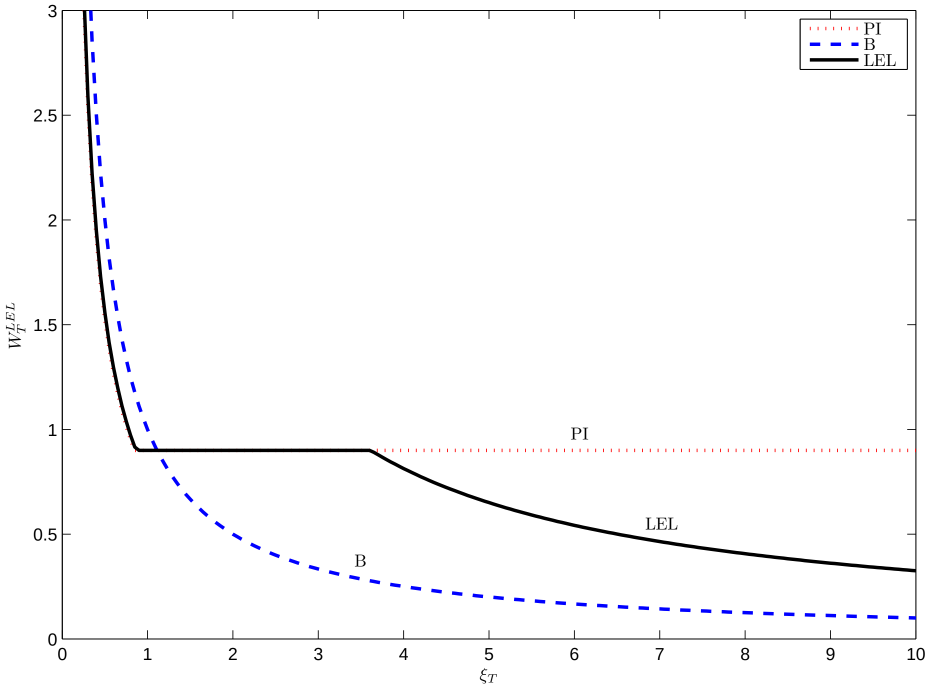

Figure 5: Optimal horizon wealth of three types of agents. The figure plots the optimal horizon wealth of the LEL agent (solid line), the benchmark agent (dashed line) and the portfolio insurance agent (dotted line). The parameter values are: $\lambda = 1$, $\lambda^Q = 1.5$, $\xi_0 = 1$, $r = 0.05$, $\eta = 0.4$, $\sigma_S = 0.18$, $T = 1$, $t = 0.5$, $\gamma = 1$, $\epsilon = 0.01$, $W_0 = 1$, $\underline{W} = 0.9$.

and provides an explanation for the failure of regulation in the financial crisis.

Table 3 shows the classification of states for optimal horizon wealth under the LEL constraint for different jump parameters. The upper bound $\bar{\xi}^{VaR}$ is increasing in the jump risk premium due to the corresponding leftward shit in the distribution of ξ_T; Confronted with a fatter tail, the agent must increase the upper bound in order to limit the magnitude of the losses in bad states. On the other hand, the budget constraint drives the lower bound $\underline{\xi}^{VaR}$ to decrease, leading to larger range for the portfolio insurance strategy. Similarly to the VaR case, Pr^{LEL} declines with the jump risk premium because of the large shift in the distribution of ξ_T. Moreover, as the jump risk premium increases, both the LEL and the PI agents behave more differently from the benchmark agent in terms of the difference in the Lagrange multiplier.

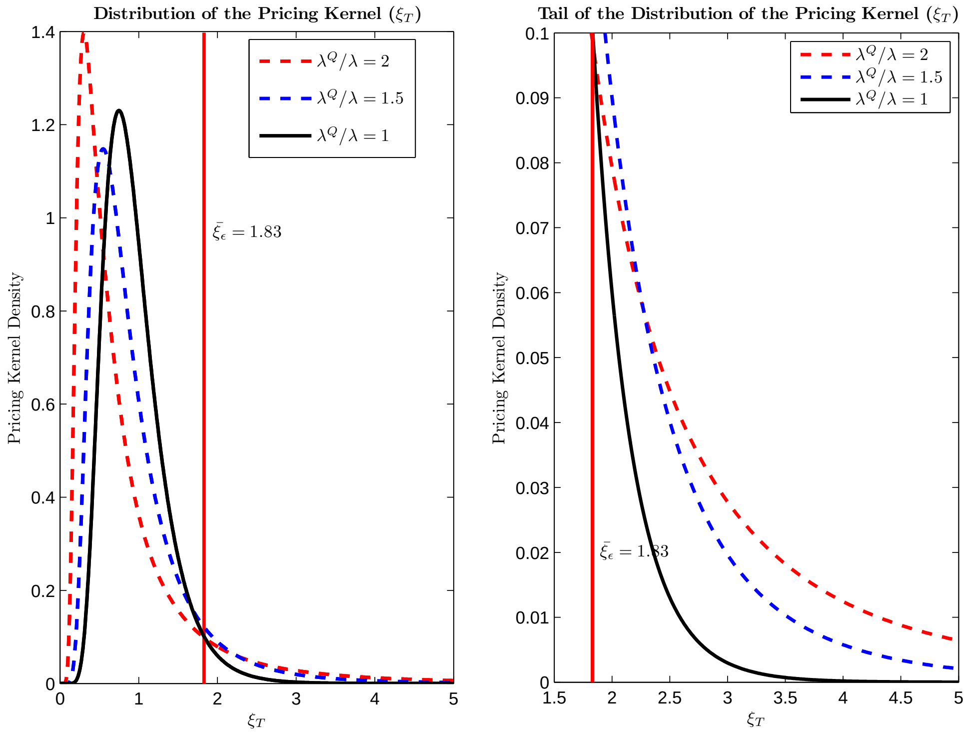

Figure 6: Distribution of the pricing kernel at horizon. The left plots the distribution of the pricing kernel at horizon for different jump parameters, while the right panel plots the upper tail of the distribution. In both panels, the black solid line is for $\lambda^Q/\lambda = 1$, the blue dashed line is for $\lambda^Q/\lambda = 1.5$ and the red dashed line is for $\lambda^Q/\lambda = 2$. The red solid line is for $\bar{\xi}^{LEL}$ in the Basak and Shapiro case. λ is fixed at 1, while λ^Q varies in different cases. Other parameter values are: $\xi_0 = 1$, $r = 0.05$, $\eta = 0.4$, $T = 1$.

4.2 Trading Strategies

Proposition 4 characterizes the optimal wealth and portfolio strategies before the horizon under the LEL constraint.

Table 3: Classification of States for Optimal Horizon Wealth Under the LEL Constraint

λ	λ^Q	$\underline{\xi}^{LEL}$	$\bar{\xi}^{LEL}$	Pr^{LEL}	Pr^{PI}	y^{LEL}	y^B	y^{PI}
1	1	0.99	1.83	35.4%	40.5%	1.13	1	1.15
1	1.5	0.86	3.61	41.2%	43.6%	1.28	1	1.31
1	2	0.71	10.52	40.7%	42.2%	1.56	1	1.60
1	3	0.67	17.78	25.8%	26.5%	1.65	1	1.69
1.5	2	0.89	2.92	40.8%	43.7%	1.24	1	1.27
1.5	3	0.66	13.72	38.8%	40.1%	1.69	1	1.74
2	3	0.78	5.76	42.9%	44.8%	1.42	1	1.46

The table shows the classification of the state of the world for optimal horizon wealth under the LEL constraint. Pr^{LEL} is the probability that the LEL agent's terminal wealth is under portfolio insurance. Pr^{PI} is the probability that the PI agent's terminal wealth is under portfolio insurance. The parameter values are: $\xi_0 = 1, r = 0.05, \eta = 0.4, T = 1$.

Proposition 4. *The time-t optimal wealth is given by*

$$\begin{aligned} W_t^{LEL} = & \frac{\mathrm{e}^{\Gamma_t}}{(y^{LEL}\xi_t)^{\frac{1}{\gamma}}} - \left\{ \frac{\mathrm{e}^{\Gamma_t}}{(y^{LEL}\xi_t)^{\frac{1}{\gamma}}} \mathrm{E}_t \left[\mathrm{e}^{\frac{1-\gamma}{\gamma}\Psi(N_t)} \mathcal{N}\left(-d_1(\underline{\xi}^{LEL}) - \frac{\Psi(N_t)}{\eta\sqrt{\tau}} \right) \right] \right. \\ & \left. - \underline{W}\mathrm{e}^{-r\tau}\mathrm{E}_t \left[\mathrm{e}^{-\Psi(N_t)} \mathcal{N}\left(-d_2(\underline{\xi}^{LEL}) - \frac{\Psi(N_t)}{\eta\sqrt{\tau}} \right) \right] \right\} \\ & + \left\{ \frac{\mathrm{e}^{\Gamma_t}}{((y^{LEL} - y_1^{LEL})\,\xi_t)^{\frac{1}{\gamma}}} \mathrm{E}_t \left[\mathrm{e}^{\frac{1-\gamma}{\gamma}\Psi(N_t)} \mathcal{N}\left(-d_1(\bar{\xi}^{LEL}) - \frac{\Psi(N_t)}{\eta\sqrt{\tau}} \right) \right] \right. \\ & \left. - \underline{W}\mathrm{e}^{-r\tau}\mathrm{E}_t \left[\mathrm{e}^{-\Psi(N_t)} \mathcal{N}\left(-d_2(\bar{\xi}^{LEL}) - \frac{\Psi(N_t)}{\eta\sqrt{\tau}} \right) \right] \right\} \end{aligned} \tag{35}$$

where $\Gamma(t)$, $d_1(x)$, $d_2(x)$ are as given in Proposition 2, (y^{LEL}, y_1^{LEL}) are as given in Proposition 3, $\underline{\xi}^{LEL} = \frac{1}{y^{LEL}\underline{W}^{\gamma}}$ and $\bar{\xi}^{LEL} = \frac{1}{\left(y^{LEL} - y_1^{LEL}\right)\underline{W}^{\gamma}}$.

The exposure of the optimal portfolio to the risk factors Z and N is given by,

$$\pi_t^{Z,LEL} = \frac{\eta}{\sigma\gamma} - \frac{\mathrm{e}^{-r\tau}\eta\underline{W}}{\sigma\gamma W_t^{LEL}} \mathrm{E}_t \left[\mathrm{e}^{-\Psi(N_t)} \left(\mathcal{N}\left(-d_2(\underline{\xi}^{LEL}) - \frac{\Psi(N_t)}{\eta\sqrt{\tau}} \right) - \mathcal{N}\left(-d_2(\bar{\xi}^{LEL}) - \frac{\Psi(N_t)}{\eta\sqrt{\tau}} \right) \right) \right], \tag{36}$$

$$\pi_t^{N,LEL} = \frac{\sigma\left(1 - \frac{\lambda^Q}{\lambda}\right)}{\mu\eta} \pi_t^{Z,LEL}. \tag{37}$$

In the benchmark case, the exposure of the optimal portfolio to the risk factor Z is,

$$\pi_t^B = \frac{\eta}{\gamma\sigma}. \tag{38}$$

Let $q_t^{LEL} = \pi_t^{Z,LEL}/\pi_t^B$. Then q_t^{LEL} is,

$$q_t^{LEL} = 1 - \frac{\mathrm{e}^{-r\tau}\underline{W}}{W_t^{LEL}}\mathrm{E}_t\left[\mathrm{e}^{-\Psi(N_t)}\left(\mathcal{N}\left(-d_2(\underline{\xi}^{LEL}) - \frac{\Psi(N_t)}{\eta\sqrt{\tau}}\right) - \mathcal{N}\left(-d_2(\bar{\xi}^{LEL}) - \frac{\Psi(N_t)}{\eta\sqrt{\tau}}\right)\right)\right]. \tag{39}$$

The exposure to the risk factor Z relative to the benchmark is bounded below and above: $0 \leq q_t^{LEL} \leq 1$, and

$$\lim_{\xi_t \to 0} q_t^{LEL} = \lim_{\xi_t \to \infty} q_t^{LEL} = 1. \tag{40}$$

Figure 7 depicts the optimal wealth of the LEL agent, the B agent and the PI agent at time t. As in the VaR case, the optimal prehorizon wealth of the VaR agent acts similarly to the benchmark agent's wealth for low and high values of ξ_t but moves towards the portfolio insurer's wealth for intermediate values of ξ_t. As the jump risk increases, the LEL agent attempts to insure increasingly more states and increase wealth in the higher tail of the xi_t distribution at the expense of reduction in wealth in good states. This implies that the jump risk drives the LEL agent to redistribute her asset allocation towards the bad states of the world and behave more similarly to the PI agent.

Figure 8 illustrates the optimal time-t equity exposure of the LEL agent, the B agent and the PI agent relative to the B agent's equity exposure. In the two extremes, the LEL agents acts similarly to the B agent and invests aggressively in the stock. In between, however, she first reduces exposure to the risky asset as the PI agent in order to fully insure against the intermediate states. Then, as the states worsen, she begins to leave states partially insured and tend back toward the benchmark behavior. It is important to note that unlike the VaR agent, the LEL agent has no incentive to gamble abound the upper bound and never takes a larger equity exposure than the B agent because of the limit on the wealth losses. Therefore, the LEL constraint turns out to remedy the shortcomings of the VaR constraint. The right panel

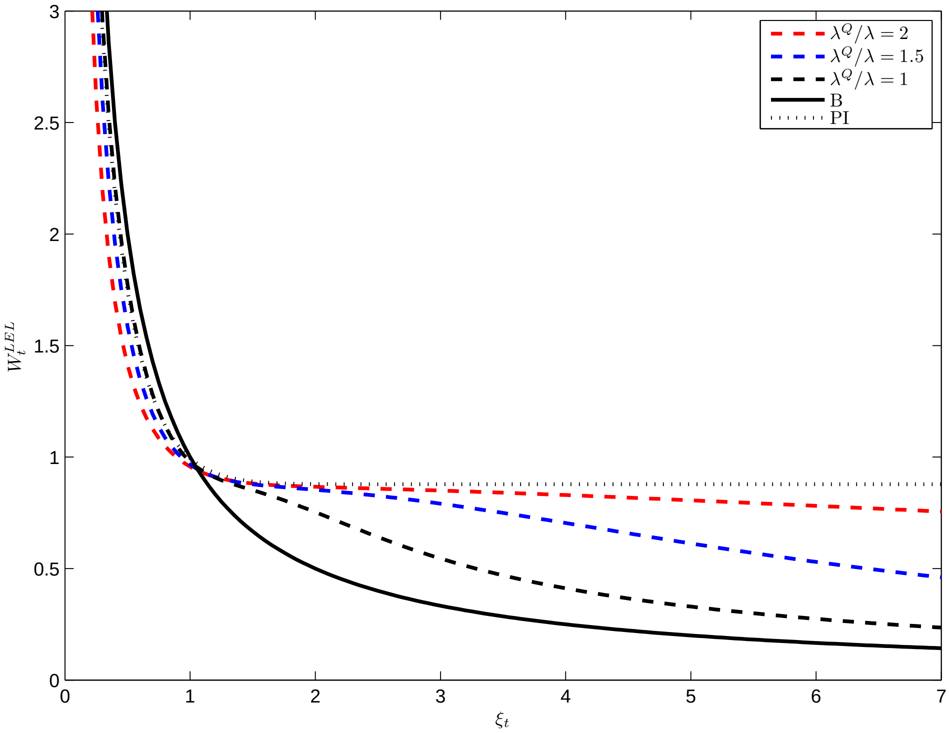

Figure 7: Optimal time-t wealth of different types of agents. The figure plots the optimal time-t wealth of the LEL agent, the benchmark agent and the portfolio insurance agent. The black solid line is for $\lambda^Q/\lambda = 1$, the blue dashed line is for $\lambda^Q/\lambda = 1.5$ and the red dashed line is for $\lambda^Q/\lambda = 2$. The solid line is for the benchmark case and the dotted line is for the portfolio insurance case. λ is fixed at 1, while λ^Q varies in different cases. Other parameter values are: $\xi_0 = 1$, $r = 0.05$, $\eta = 0.4$, $\sigma_S = 18$, $T = 1$, $t = 0.5$, $\gamma = 1$, $\epsilon = 0.01$, $W_0 = 1$, $\underline{W} = 0.9$.

shows the optimal prehorizon equity exposure without the jump risk for different levels of loss tolerance. Obviously, the higher jump risk has similar effects to tighter LEL constraint on the exposure to the stock: as the jump size of ξ_t rises, the LEL is confronted with higher probability of occurrence of bad states and has to shrink the bad-states region, which is associated with lower equity exposure in the intermediate region. As a consequence, she moves towards the PI policy, which is similar to the effect of lower ϵ.

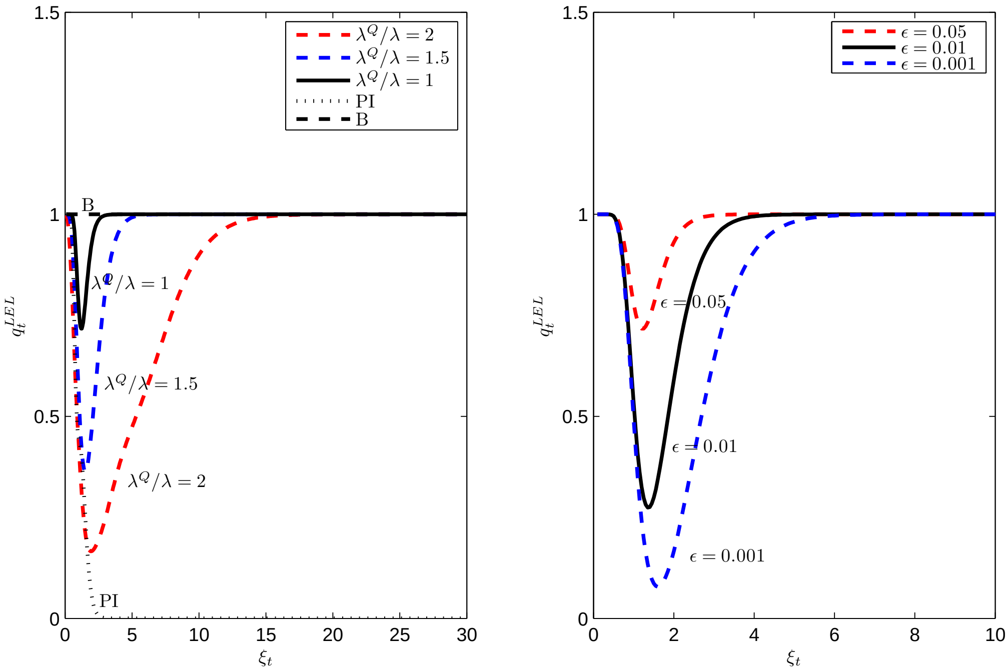

Figure 8: Optimal time-t equity exposure of different types of agents. The left panel plots the optimal time-t equity exposure of the LEL agent, the benchmark agent and the portfolio insurance agent. The right panel plots the optimal pre-horizon risk exposure of the LEL agent in the Basak and Shapiro case for different α for comparison purpose. In the left panel, the solid line is for $\lambda^Q/\lambda = 1$, the blue dashed line is for $\lambda^Q/\lambda = 1.5$ and the red dashed line is for $\lambda^Q/\lambda = 2$. The black dashed line is for the benchmark case and the dotted line is for the portfolio insurance case. In the right panel, the solid line is for $\epsilon = 0.01$, the blue dashed line is for $\epsilon = 0.001$ and the red dashed line is for $\epsilon = 0.05$. λ is fixed at 1, while λ^Q varies in different cases. Other parameter values are: $\xi_0 = 1$, $r = 0.05$, $\eta = 0.4$, $\sigma_S = 18$, $T = 1$, $t = 0.5$, $\gamma = 1$, $\alpha = 0.01$, $W_0 = 1$, $\underline{W} = 0.9$.

5 Optimization under the CVaR Constraint

The portfolio optimization problem of the CVaR risk manager can be formulated as follows,

$$\max_{W_T} E\left[\frac{W_T^{1-\gamma}}{1-\gamma}\right] \tag{41}$$

$$\text{subject to} \quad E[\xi_T W_T] \leqslant \xi_0 W_0, \tag{42}$$

$$E\left[(\underline{W} - W_T)1_{\{W_T \leqslant \underline{W}\}}\right] \leqslant \delta. \tag{43}$$

where $\delta \geq 0$ is a constant. Note that (43) also nests both the B-case and the PI-case, which correspond to $\delta = \infty$ and $\delta = 0$ respectively.

Proposition 5 characterizes the optimal terminal wealth under the CVaR constraint.

Proposition 5. *The time-T optimal wealth of the CVaR agent is*

$$W_T^{CVaR} = \begin{cases} \left(y^{CVaR}\xi_T\right)^{-\frac{1}{\gamma}} & \text{if } \xi_T < \underline{\xi}^{CVaR}, \\ \underline{W} & \text{if } \underline{\xi}^{CVaR} \leqslant \xi_T < \bar{\xi}^{CVaR}, \\ \left(y^{CVaR}\xi_T - y_1^{CVaR}\right)^{-\frac{1}{\gamma}} & \text{if } \xi_T \geqslant \bar{\xi}^{CVaR}. \end{cases} \tag{44}$$

where $\underline{\xi}^{CVaR} = \frac{\underline{W}^{-\gamma}}{y^{CVaR}}$, $\bar{\xi}^{CVaR} = \frac{\underline{W}^{-\gamma}+y_1^{CVaR}}{y^{CVaR}}$, *and* $y^{CVaR} \geqslant 0$, $y_1^{CVaR} \geqslant 0$ *are two Lagrange multipliers of the budget constraint and CVaR constraint respectively and solve the following system:*

$$\begin{cases} E[\xi_T W^{CVaR}(T; y^{CVaR}, y_1^{CVaR})] = \xi_0 W_0, \\ E\left[(\underline{W} - W^{CVaR}(T; y^{CVaR}, y_1^{CVaR}))1_{\left\{W^{CVaR}(T;y^{CVaR},y_1^{CVaR})\leqslant \underline{W}\right\}}\right] = \delta \text{ or } y_1^{CVaR} = 0. \end{cases} \tag{45}$$

The CVaR constraint in (43) *is binding if, and only if,* $\underline{\xi}^{CVaR} < \bar{\xi}^{CVaR}$. *Moreover, the Lagrange multiplier* y^{CVaR} *is decreasing in* δ, $y^{CVaR} \in \left[y^B, y^{PI}\right]$, *and* $y^{CVaR} - y_1^{CVaR} \leqslant y^B$.

Proposition 6 characterizes the optimal wealth and portfolio strategies before the horizon under the CVaR constraint.

Proposition 6. *The time-t optimal wealth is given by*

$$\begin{aligned} W_t^{CVaR} = {} & \frac{\mathrm{e}^{\Gamma_t}}{(y^{CVaR}\xi_t)^{\frac{1}{\gamma}}} - \left\{ \frac{\mathrm{e}^{\Gamma_t}}{(y^{CVaR}\xi_t)^{\frac{1}{\gamma}}} \mathrm{E}_t\left[\mathrm{e}^{\frac{1-\gamma}{\gamma}\Psi(N_t)}\mathcal{N}\left(-d_1(\underline{\xi}^{CVaR}) - \frac{\Psi(N_t)}{\eta\sqrt{\tau}}\right)\right] \right. \\ & \left. - \underline{W}\mathrm{e}^{-r\tau}\mathrm{E}_t\left[\mathrm{e}^{-\Psi(N_t)}\mathcal{N}\left(-d_2(\underline{\xi}^{CVaR}) - \frac{\Psi(N_t)}{\eta\sqrt{\tau}}\right)\right]\right\} \\ & + \left\{\frac{1}{\xi_t}\mathrm{E}_t\left[\int_{\ln \bar{\xi}^{CVaR}}^{+\infty} \mathrm{e}^{\ln \xi_T - \frac{1}{\gamma}\ln(y^{CVaR}\xi_T - y_1^{CVaR})}\frac{1}{\eta\sqrt{2\pi\tau}}\mathrm{e}^{-\frac{(\ln \xi_T - A)^2}{2\eta^2\tau}} d\ln\xi_T\right] \right. \\ & \left. - \underline{W}\mathrm{e}^{-r\tau}\mathrm{E}_t\left[\mathrm{e}^{-\Psi(N_t)}\mathcal{N}\left(-d_2(\bar{\xi}^{CVaR}) - \frac{\Psi(N_t)}{\eta\sqrt{\tau}}\right)\right]\right\} \end{aligned} \tag{46}$$

where $\Gamma(t)$, $d_1(x)$, $d_2(x)$ *are as given in Proposition 2,* A *is as given in Appendix (Proof of Proposition 2,* $A = \ln \xi_t - (r + \frac{1}{2}\eta^2)\tau - \Psi(N_t)$*),* y^{CVaR}, y_1^{CVaR}, $\underline{\xi}^{CVaR}$ *and* $\bar{\xi}^{CVaR}$ *and are as given in Proposition 5.*

Under the CVaR constraint, neither W_t^{CVaR} *nor* $\frac{dW_t^{CVaR}}{d\xi_t}$ *can have analytical solutions. However, in our numerical approximation,* $\frac{dW_t^{CVaR}}{d\xi_t}$ *can be formulated as follows,*

$$\frac{dW_t^{CVaR}}{d\xi_t} \approx \frac{W_{t,\xi_t+\triangle\xi_t}^{CVaR} - W_{t,\xi_t-\triangle\xi_t}^{CVaR}}{2\triangle\xi_t}, \tag{47}$$

where $W_{t,\xi_t+\triangle\xi_t}^{CVaR}$ $\left(W_{t,\xi_t-\triangle\xi_t}^{CVaR}\right)$ *is the time-t optimal wealth when the pricing kernel is the value* $\xi_t + \triangle\xi_t$ $(\xi_t - \triangle\xi_t)$ *, holding other values constant.*

The exposure of the optimal portfolio to the risk factors Z *and* N *is given by,*

$$\begin{aligned} \pi_t^{Z,CVaR} &= -\frac{\eta}{\sigma}\frac{\xi_{t-}}{W_{t-}}\frac{dW_t^{CVaR}}{d\xi_t} && (48)\\ &= -\frac{\eta}{\sigma}\frac{\xi_t}{W_t^{CVaR}}\frac{dW_t^{CVaR}}{d\xi_t} && (49)\\ &= -\frac{\eta}{\sigma}\frac{\xi_t}{W_t^{CVaR}}\frac{W_{t,\xi_t+\triangle\xi_t}^{CVaR} - W_{t,\xi_t-\triangle\xi_t}^{CVaR}}{2\triangle\xi_t}, && (50) \end{aligned}$$

$$\pi_t^{N,CVaR} = \frac{\sigma\left(1 - \frac{\lambda^Q}{\lambda}\right)}{\mu\eta}\pi_t^{Z,CVaR}. \tag{51}$$

In the benchmark case, the exposure of the optimal portfolio to the risk factor Z *is,*

$$\pi_t^B = \frac{\eta}{\gamma\sigma}. \tag{52}$$

Let $q_t^{CVaR} = \pi_t^{Z,CVaR}/\pi_t^B$*. Then* q_t^{CVaR} *is,*

$$q_t^{CVaR} = -\frac{1}{\gamma}\frac{\xi_t}{W_t^{CVaR}}\frac{W_{t,\xi_t+\triangle\xi_t}^{CVaR} - W_{t,\xi_t-\triangle\xi_t}^{CVaR}}{2\triangle\xi_t}. \tag{53}$$

Table 4: Classification of States for Optimal Horizon Wealth Under the CVaR Constraint

λ	λ^Q	$\underline{\xi}^{CVaR}$	$\bar{\xi}^{CVaR}$	Pr^{CVaR}	Pr^{PI}	y^{CVaR}	y^B	y^{PI}
1	1	1.01	1.68	31.3%	40.5%	1.10	1	1.15
1	1.5	0.92	2.60	34.9%	43.6%	1.20	1	1.31
1	2	0.84	4.51	31.5%	42.2%	1.32	1	1.60
1	3	0.86	7.07	19.9%	26.5%	1.30	1	1.69
1.5	2	0.94	2.31	35.2%	43.7%	1.18	1	1.27
1.5	3	0.82	5.49	29.3%	40.1%	1.36	1	1.74
2	3	0.87	3.38	35.3%	44.8%	1.28	1	1.46

The table shows the classification of the state of the world for optimal horizon wealth under the CVaR constraint. Pr^{CVaR} is the probability that the CVaR agent's terminal wealth is under portfolio insurance. Pr^{PI} is the probability that the PI agent's terminal wealth is under portfolio insurance. The parameter values are: $\xi_0 = 1$, $r = 0.05$, $\eta = 0.4$, $T = 1$.

6 Conclusion

This paper studies optimal portfolio and wealth policies of an institutional investor under the VaR constraint in an economy under jump diffusion. In our framework, the investor can access not only the bond and stock markets but also to the option market to manage the jump risk exposure and the diffusion risk exposure. The results are encouraging in several aspects. First, we document that underestimating jump risk factor could be the cause of failure to satisfy the VaR and LEL constraints in the recent financial crisis for many financial institutions. Second, the introduction of the jump risk factor drives the institutional investor to behave like the portfolio insurance manager, alleviating the problem with VaR identified by Basak and Shapiro (2001) that VaR risk manager incurs larger losses than non risk manager in worst scenarios. Third, unlike the VaR agent, the LEL agent has no incentive to gamble abound the upper bound and never takes a larger equity exposure than the B agent because of the limit on the wealth losses. Therefore, the LEL constraint turns out to remedy the shortcomings of the VaR constraint.

To our best knowledge, this paper is the first to adopt pricing kernel with jumps in modeling institutional investors as expected utility maximizers, who must comply with a VaR constraint or a LEL constraint imposed at some horizon. As a result, this paper provides a potentially useful way to evaluate VaR risk management and LEL risk management during the financial crisis. The results here can be used both as benchmarks for models of VaR and as directions

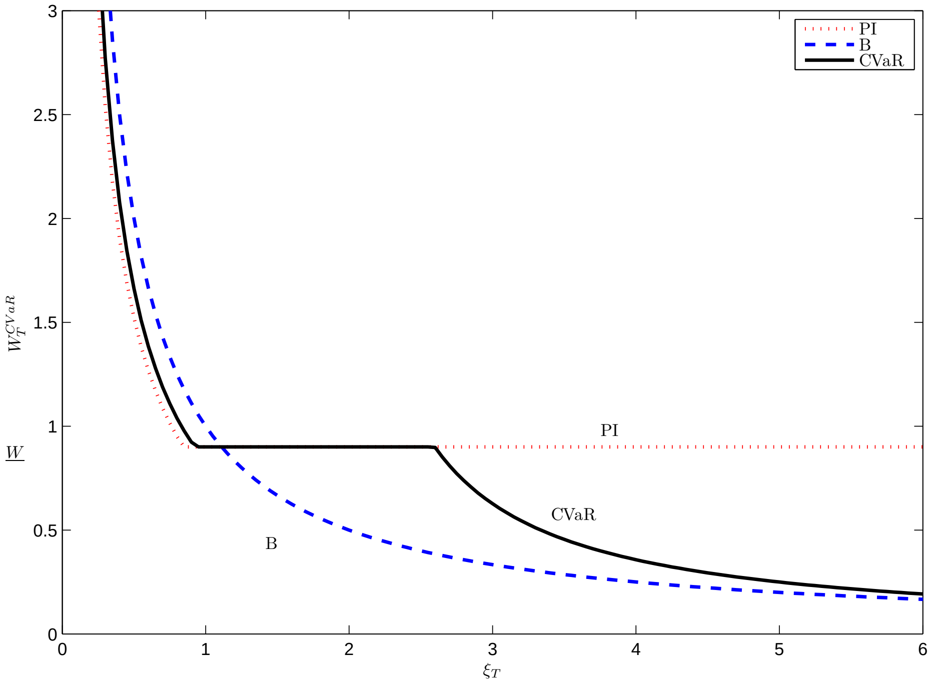

Figure 9: Optimal horizon wealth of three types of agents. The figure plots the optimal horizon wealth of the CVaR agent (solid line), the benchmark agent (dashed line) and the portfolio insurance agent (dotted line). The parameter values are: $\lambda = 1$, $\lambda^Q = 1.5$, $\xi_0 = 1$, $r = 0.05$, $\eta = 0.4$, $\sigma_S = 0.18$, $T = 1$, $t = 0.5$, $\gamma = 1$, $\epsilon = 0.01$, $W_0 = 1$, $\underline{W} = 0.9$.

for future research. For example, there appears to be some parameter instability as the jump risk premium change, our analysis may pave the way to modeling model uncertainty.

A Proof of Proposition 1

As we completely follow Basak and Shapiro (2001) in deriving Proposition 1, one can see the Proof of Proposition 1 in the Appendix of Basak and Shapiro (2001) for reference.

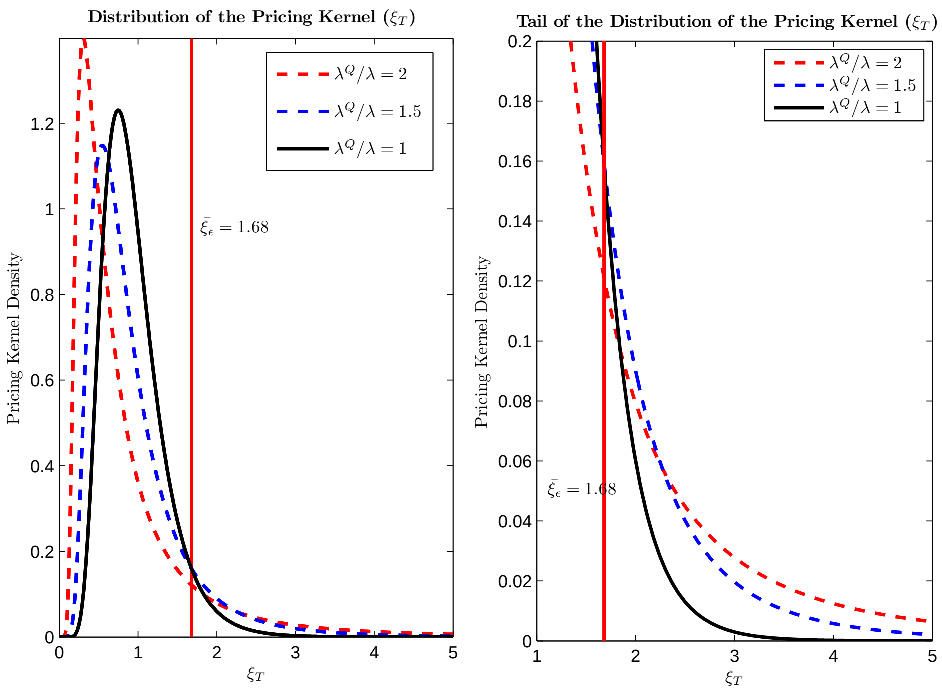

Figure 10: Distribution of the pricing kernel at horizon. The left plots the distribution of the pricing kernel at horizon for different jump parameters, while the right panel plots the upper tail of the distribution. In both panels, the black solid line is for $\lambda^Q/\lambda = 1$, the blue dashed line is for $\lambda^Q/\lambda = 1.5$ and the red dashed line is for $\lambda^Q/\lambda = 2$. The red solid line is for $\bar{\xi}$ in the Basak and Shapiro case. λ is fixed at 1, while λ^Q varies in different cases. Other parameter values are: $\xi_0 = 1$, $r = 0.05$, $\eta = 0.4$, $T = 1$.

B Proof of Proposition 2

With complete market assumption, Itô's lemma implies that $\xi_t W_t$ is a martingale:

$$
\begin{aligned}
W_t^{VaR} &= \mathrm{E}_t\left[\frac{\xi_T}{\xi_t} W_T^{VaR}\right] \\
&= \mathrm{E}_t\left[\frac{\xi_T}{\xi_t} I(y\xi_T) | \xi_T < \underline{\xi}^{VaR}\right] + \mathrm{E}_t\left[\frac{\xi_T}{\xi_t} \underline{W} | \underline{\xi} < \xi_T < \bar{\xi}^{VaR}\right] + \mathrm{E}_t\left[\frac{\xi_T}{\xi_t} I(y\xi_T) | \xi_T > \bar{\xi}^{VaR}\right]
\end{aligned}
\tag{54}
$$

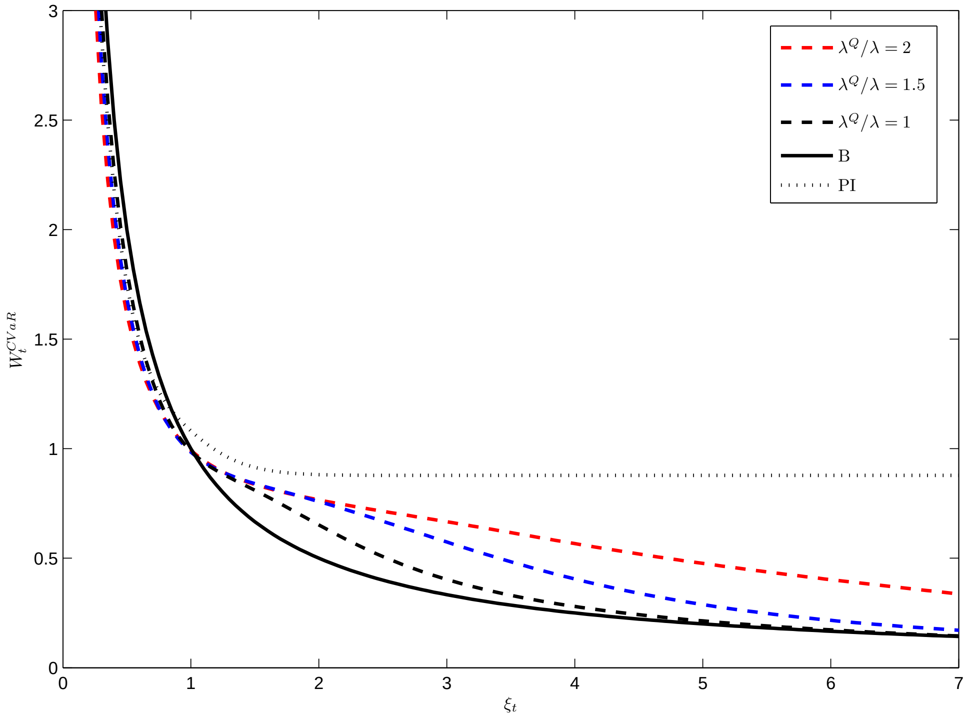

Figure 11: Optimal time-t wealth of different types of agents. The figure plots the optimal time-t wealth of the CVaR agent, the benchmark agent and the portfolio insurance agent. The black solid line is for $\lambda^Q/\lambda = 1$, the blue dashed line is for $\lambda^Q/\lambda = 1.5$ and the red dashed line is for $\lambda^Q/\lambda = 2$. The solid line is for the benchmark case and the dotted line is for the portfolio insurance case. λ is fixed at 1, while λ^Q varies in different cases. Other parameter values are: $\xi_0 = 1$, $r = 0.05$, $\eta = 0.4$, $\sigma_S = 18$, $T = 1$, $t = 0.5$, $\gamma = 1$, $\epsilon = 0.01$, $W_0 = 1$, $\underline{W} = 0.9$.

We compute each term in (54) separately,

$$\begin{aligned} \mathrm{E}_t\left[\frac{\xi_T}{\xi_t} I(y\xi_T)|\xi_T < \underline{\xi}^{VaR}\right] &= \mathrm{E}_t\left[\frac{\xi_T}{\xi_t}\frac{1}{(y\xi_T)^{\frac{1}{\gamma}}}|\xi_T < \underline{\xi}^{VaR}\right] \\ &= \frac{1}{y^{\frac{1}{\gamma}}\xi_t}\mathrm{E}_t\left[\mathrm{E}_t\left[\xi_T^{1-\frac{1}{\gamma}}|\xi_T < \underline{\xi}^{VaR}, \sigma(N_T - N_t)\right]\right] \qquad (55) \end{aligned}$$

where the conditioning σ-algebra $\sigma(N_T - N_t)$ is the one generated by the random variable $(N_T - N_t)$. To avoid confusion with the volatility parameter σ, we simply write it as $(N_T - N_t)$ in what follows.

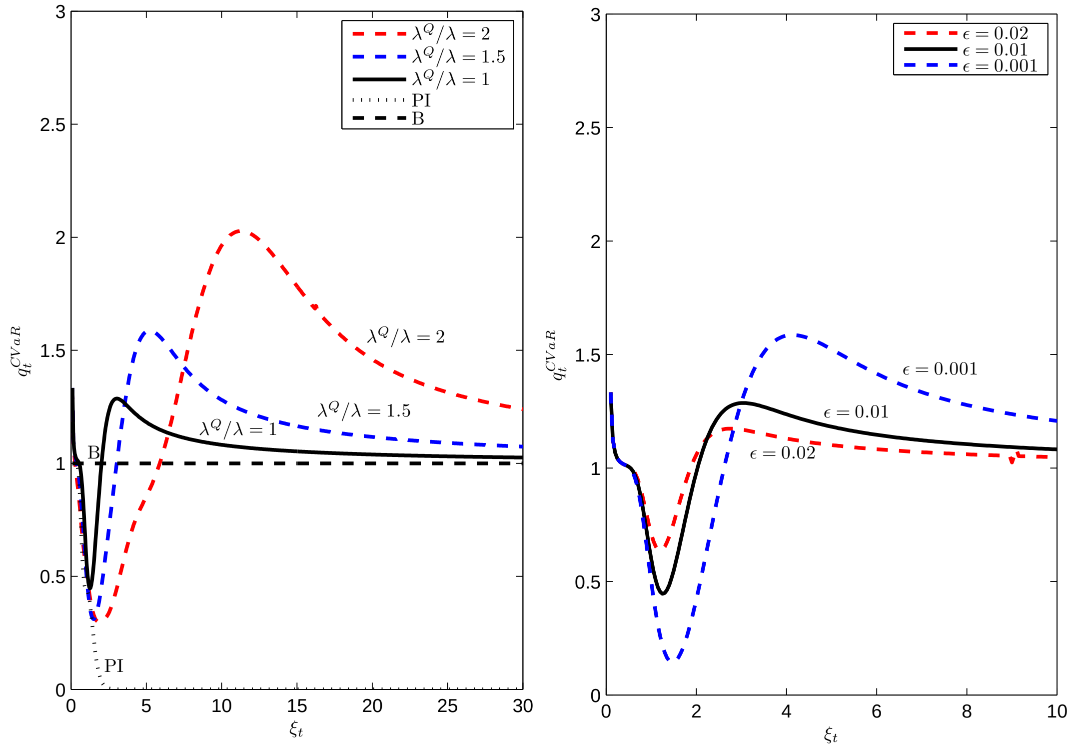

Figure 12: Optimal time-t equity exposure of different types of agents. The left panel plots the optimal time-t equity exposure of the CVaR agent, the benchmark agent and the portfolio insurance agent. The right panel plots the optimal pre-horizon risk exposure of the CVaR agent in the Basak and Shapiro case for different α for comparison purpose. In the left panel, the solid line is for $\lambda^Q/\lambda = 1$, the blue dashed line is for $\lambda^Q/\lambda = 1.5$ and the red dashed line is for $\lambda^Q/\lambda = 2$. The black dashed line is for the benchmark case and the dotted line is for the portfolio insurance case. In the right panel, the solid line is for $\epsilon = 0.01$, the blue dashed line is for $\epsilon = 0.001$ and the red dashed line is for $\epsilon = 0.05$. λ is fixed at 1, while λ^Q varies in different cases. Other parameter values are: $\xi_0 = 1$, $r = 0.05$, $\eta = 0.4$, $\sigma_S = 18$, $T = 1$, $t = 0.5$, $\gamma = 1$, $\alpha = 0.01$, $W_0 = 1$, $\underline{W} = 0.9$.

It is easy to see $\ln \xi_T$ follows normal distribution conditional on both $\mathcal{F}_t$ and $(N_T - N_t)$,

$$\ln \xi_T | \mathcal{F}_t, (N_T - N_t) \sim \mathcal{N}(\ln \xi_t - (r + \frac{1}{2}\eta^2)\tau - \Psi(N_t), \eta^2 \tau). \tag{56}$$

Let $A = \ln \xi_t - (r + \frac{1}{2}\eta^2)\tau - \Psi(N_t)$. Then, (56) implies

$$\begin{aligned}
& \mathrm{E}_t \left[\xi_T^{1-\frac{1}{\gamma}} | \xi_T < \underline{\xi}^{VaR}, (N_T - N_t) \right] \\
&= \int_{-\infty}^{\ln \underline{\xi}^{VaR}} \mathrm{e}^{\frac{\gamma-1}{\gamma} \ln \xi_T} \frac{1}{\sqrt{2\pi\tau}\eta} \mathrm{e}^{-\frac{(\ln \xi_T - A)^2}{2\eta^2 \tau}} \mathrm{d} \ln \xi_T \\
&= \exp\left(\left(\frac{\gamma-1}{\gamma} \right)^2 \frac{\eta^2}{2}\tau + \frac{\gamma-1}{\gamma} A \right) \int_{-\infty}^{\ln \underline{\xi}^{VaR}} \frac{1}{\sqrt{2\pi\tau}\eta} \mathrm{e}^{-\frac{\left(\ln \xi_T - \left(A + \eta^2\left(1-\frac{1}{\gamma}\right)\tau\right)\right)^2}{2\eta^2\tau}} \mathrm{d} \ln \xi_T \\
&= \frac{\mathrm{e}^{\Gamma_t}}{\xi_t^{\frac{1}{\gamma}-1}} \mathrm{e}^{\frac{1-\gamma}{\gamma}\Psi(N_t)} \mathcal{N}\left(d_1(\underline{\xi}^{VaR}) + \frac{\Psi(N_t)}{\eta\sqrt{\tau}} \right) \qquad (57)
\end{aligned}$$

Substituting (57) into (54) yields,

$$\begin{aligned}
\mathrm{E}_t \left[\frac{\xi_T}{\xi_t} I(y\xi_T) | \xi_T < \underline{\xi}^{VaR} \right] &= \frac{1}{y^{\frac{1}{\gamma}} \xi_t} \mathrm{E}_t \left[\frac{\mathrm{e}^{\Gamma_t}}{\xi_t^{\frac{1}{\gamma}-1}} \mathrm{e}^{\frac{1-\gamma}{\gamma}\Psi(N_t)} \mathcal{N}\left(d_1(\underline{\xi}^{VaR}) + \frac{\Psi(N_t)}{\eta\sqrt{\tau}} \right) \right] \\
&\approx \frac{\mathrm{e}^{\Gamma_t}}{(y\xi_t)^{\frac{1}{\gamma}}} \left\{ 1 - \mathrm{E}_t \left[\mathrm{e}^{\frac{1-\gamma}{\gamma}\Psi(N_t)} \mathcal{N}\left(-d_1(\underline{\xi}^{VaR}) - \frac{\Psi(N_t)}{\eta\sqrt{\tau}} \right) \right] \right\}, \\
& \qquad (58)
\end{aligned}$$

where the approximation follows from the second order Taylor approximation. One can easily calculate the remaining two terms in (54) in a similar fashion,

$$\begin{aligned}
\mathrm{E}_t \left[\frac{\xi_T}{\xi_t} \underline{W} | \underline{\xi}^{VaR} < \xi_T < \bar{\xi}^{VaR} \right] &= \mathrm{e}^{-r\tau} \underline{W} \mathrm{E}_t \left[\mathrm{e}^{-\Psi(N_t)} \left(-\mathcal{N}\left(d_2(\underline{\xi}^{VaR}) + \frac{\Psi(N_t)}{\eta\sqrt{\tau}} \right) \right. \right. \\
& \qquad \left. \left. + \mathcal{N}\left(d_2(\bar{\xi}^{VaR}) + \frac{\Psi(N_t)}{\eta\sqrt{\tau}} \right) \right) \right] \qquad (59) \\
\mathrm{E}_t \left[\frac{\xi_T}{\xi_t} I(y\xi_T) | \xi_T > \bar{\xi}^{VaR} \right] &= \frac{\mathrm{e}^{\Gamma_t}}{(y\xi_t)^{\frac{1}{\gamma}}} \mathrm{E}_t \left[\mathrm{e}^{\frac{1-\gamma}{\gamma}\Psi(N_t)} \mathcal{N}\left(-d_1(\bar{\xi}^{VaR}) - \frac{\Psi(N_t)}{\eta\sqrt{\tau}} \right) \right] \qquad (60)
\end{aligned}$$

Summing up (58), (59), (60), we obtain (54).

Applying Itô's lemma to (54), we can easily get (17) and (18).

C Proof of Proposition 3

The proof is as of proposition 1, except with y^{VaR}, $\underline{\xi}^{VaR}$, and $\bar{\xi}^{VaR}$ replaced appropriately by y^{LEL}, $\underline{\xi}^{LEL}$, and $\bar{\xi}^{LEL}$.

D Proof of Proposition 4

The proof of W_t^{LEL} is as of the proof of W_t^{VaR} in proposition 2, except with y^{VaR}, $\underline{\xi}^{VaR}$, and $\bar{\xi}^{VaR}$ replaced appropriately by y^{LEL}, $\underline{\xi}^{LEL}$, and $\bar{\xi}^{LEL}$.

Applying Itô's lemma to (35), we can get

$$
\begin{aligned}
\frac{dW_t^{LEL}}{d\xi_t} &= \frac{\mathrm{e}^{\Gamma_t}}{(y^{LEL}\xi_t)^{\frac{1}{\gamma}}}(-\frac{1}{\gamma})(\xi_t)^{-1}\left\{1-\mathrm{E}_t\left[\mathrm{e}^{\frac{1-\gamma}{\gamma}\Psi(N_t)}\mathcal{N}\left(-d_1(\underline{\xi}^{LEL})-\frac{\Psi(N_t)}{\eta\sqrt{\tau}}\right)\right]\right\} \\
&+\frac{\mathrm{e}^{\Gamma_t}}{((y^{LEL}-y_1^{LEL})\,\xi_t)^{\frac{1}{\gamma}}}(-\frac{1}{\gamma})(\xi_t)^{-1}\mathrm{E}_t\left[\mathrm{e}^{\frac{1-\gamma}{\gamma}\Psi(N_t)}\mathcal{N}\left(-d_1(\bar{\xi}^{LEL})-\frac{\Psi(N_t)}{\eta\sqrt{\tau}}\right)\right] \\
&+\frac{\mathrm{e}^{\Gamma_t}}{(y^{LEL}\xi_t)^{\frac{1}{\gamma}}}(-1)\left(\eta\sqrt{\tau}\right)^{-1}(\xi_t)^{-1}\mathrm{E}_t\left[\mathrm{e}^{\frac{1-\gamma}{\gamma}\Psi(N_t)}\phi\left(-d_1(\underline{\xi}^{LEL})-\frac{\Psi(N_t)}{\eta\sqrt{\tau}}\right)\right] \\
&+\frac{\mathrm{e}^{\Gamma_t}}{((y^{LEL}-y_1^{LEL})\,\xi_t)^{\frac{1}{\gamma}}}\left(\eta\sqrt{\tau}\right)^{-1}(\xi_t)^{-1}\mathrm{E}_t\left[\mathrm{e}^{\frac{1-\gamma}{\gamma}\Psi(N_t)}\phi\left(-d_1(\bar{\xi}^{LEL})-\frac{\Psi(N_t)}{\eta\sqrt{\tau}}\right)\right] \\
&+\underline{W}\mathrm{e}^{-r\tau}\left(\eta\sqrt{\tau}\right)^{-1}(\xi_t)^{-1}\mathrm{E}_t\left[\mathrm{e}^{-\Psi(N_t)}\phi\left(-d_2(\underline{\xi}^{LEL})-\frac{\Psi(N_t)}{\eta\sqrt{\tau}}\right)\right] \\
&-\underline{W}\mathrm{e}^{-r\tau}\left(\eta\sqrt{\tau}\right)^{-1}(\xi_t)^{-1}\mathrm{E}_t\left[\mathrm{e}^{-\Psi(N_t)}\phi\left(-d_2(\bar{\xi}^{LEL})-\frac{\Psi(N_t)}{\eta\sqrt{\tau}}\right)\right] \qquad (61)\\
&= \frac{\mathrm{e}^{\Gamma_t}}{(y^{LEL}\xi_t)^{\frac{1}{\gamma}}}(-\frac{1}{\gamma})(\xi_t)^{-1}\left\{1-\mathrm{E}_t\left[\mathrm{e}^{\frac{1-\gamma}{\gamma}\Psi(N_t)}\mathcal{N}\left(-d_1(\underline{\xi}^{LEL})-\frac{\Psi(N_t)}{\eta\sqrt{\tau}}\right)\right]\right\} \\
&+\frac{\mathrm{e}^{\Gamma_t}}{((y^{LEL}-y_1^{LEL})\,\xi_t)^{\frac{1}{\gamma}}}(-\frac{1}{\gamma})(\xi_t)^{-1}\mathrm{E}_t\left[\mathrm{e}^{\frac{1-\gamma}{\gamma}\Psi(N_t)}\mathcal{N}\left(-d_1(\bar{\xi}^{LEL})-\frac{\Psi(N_t)}{\eta\sqrt{\tau}}\right)\right]. \qquad (62)
\end{aligned}
$$

The exposure of the optimal portfolio to the risk factor Z and N is given by,

$$\begin{aligned}
\pi_t^{Z,LEL} &= -\frac{\eta}{\sigma}\frac{\xi_t}{W_t^{LEL}}\frac{dW_t^{LEL}}{d\xi_t} \\
&= \frac{\eta}{\gamma\sigma}\frac{1}{W_t^{LEL}}\frac{\mathrm{e}^{\Gamma_t}}{(y^{LEL}\xi_t)^{\frac{1}{\gamma}}}\left\{1-\mathrm{E}_t\left[\mathrm{e}^{\frac{1-\gamma}{\gamma}\Psi(N_t)}\mathcal{N}\left(-d_1(\underline{\xi}^{LEL})-\frac{\Psi(N_t)}{\eta\sqrt{\tau}}\right)\right]\right\} \\
&\quad+\frac{\eta}{\gamma\sigma}\frac{1}{W_t^{LEL}}\frac{\mathrm{e}^{\Gamma_t}}{\left(\left(y^{LEL}-y_1^{LEL}\right)\xi_t\right)^{\frac{1}{\gamma}}}\mathrm{E}_t\left[\mathrm{e}^{\frac{1-\gamma}{\gamma}\Psi(N_t)}\mathcal{N}\left(-d_1(\bar{\xi}^{LEL})-\frac{\Psi(N_t)}{\eta\sqrt{\tau}}\right)\right] \\
&= \frac{\eta}{\sigma\gamma}-\frac{\mathrm{e}^{-r\tau}\eta\underline{W}}{\sigma\gamma W_t^{LEL}}\mathrm{E}_t\left[\mathrm{e}^{-\Psi(N_t)}\mathcal{N}\left(-d_2(\underline{\xi}^{LEL})-\frac{\Psi(N_t)}{\eta\sqrt{\tau}}\right)\right] \\
&\quad+\frac{\mathrm{e}^{-r\tau}\eta\underline{W}}{\sigma\gamma W_t^{LEL}}\mathrm{E}_t\left[\mathrm{e}^{-\Psi(N_t)}\mathcal{N}\left(-d_2(\bar{\xi}^{LEL})-\frac{\Psi(N_t)}{\eta\sqrt{\tau}}\right)\right], \qquad (63)
\end{aligned}$$

$$\pi_t^{N,LEL} = \frac{\sigma\left(1-\frac{\lambda^Q}{\lambda}\right)}{\mu\eta}\pi_t^{Z,LEL}. \qquad (64)$$

References

Ahn, Dong-Hyun, Jacob Boudoukh, Matthew Richardson, and Robert F. Whitelaw, 1999, Optimal Risk Management Using Options, *The Journal of Finance* 54, 359–375.

Basak, Suleyman, and Alexander Shapiro, 2001, Value-at-Risk-Based Risk Management : Optimal Policies and Asset Prices, 14, 371–405.

Bates, David S., 2000, Post-’87 Crash Fears in the S&P 500 Futures Option Market, *Journal of Econometrics* 94, 181–238.

Branger, Nicole, Christian Schlag, and Eva Schneider, 2008, Optimal Portfolios When Volatility Can Jump, *Journal of Banking & Finance* 32, 1087–1097.

Cuoco, D., H. He, and S. Isaenko, 2008, Optimal Dynamic Trading Strategies with Risk Limits, *Operations Research* 56, 358–368.

Driessen, J., and P. Maenhout, 2007, An Empirical Portfolio Perspective on Option Pricing Anomalies, *Review of Finance* 11, 561–603.

Kou, S.G., 2002, A Jump-Diffusion Model for Option Pricing, *Management Science* 48, 1086–1101.

Liu, Jun, and Jun Pan, 2003, Dynamic Derivative Strategies, *Journal of Financial Economics* 69, 401–430.

Pan, Jun, 2002, The Jump-Risk Premia Implicit in Options : Evidence from an Integrated Time-Series Study, *Journal of Financial Economics* 63, 3–50.

Shi, Zhen, and Bas J.M. Werker, 2012, Short-horizon regulation for long-term investors, *Journal of Banking & Finance* 36, 3227–3238.

Tan, S., 2009, The role of options in long horizon portfolio choise, *Available at SSRN 1324464*.

Zhang, J. E., H. Zhao, and E. C Chang, 2012, Equlibrium Asset and Option Pricing under Jump Diffusion, *Mathematical Finance* 22, 538–568.